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«WOULD *YOU* SOLVE THE DIRAC EQUATION?»

Ya.I.Granovsky

«ВЫ БЫ РЕШИЛИ УРАВНЕНИЕ ДИРАКА?»

Я.И.Грановский

Новгородский государственный университет имени Ярослава Мудрого, yagran1931@gmail.com

This question belongs to great physicist Richard Feynman and presents a challenge to all the physical community. I have the courage to answer the question – it contained in Part I of this paper. Two other Parts are devoted to its Analysis and Remarks.

Keywords: Dirac equation, Whittaker equation, bispinor, radial function

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Этот вопрос принадлежит великому физику Ричарду Фейнману и представляет собой проблему для всего физического сообщества. Ответ на вопрос содержится в Части I этой статьи. Две другие части посвящены его анализу и замечаниям.

Ключевые слова: уравнение Дирака, уравнение Уиттекера, биспинор, радиальная функция

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I. Solution

The 1-st order wave equation was invented in 1928 by P.A.M.Dirac [1] in the form $(\vec{\alpha}\vec{p}+\beta m)\Psi=E\Psi$ and was improved by W. Pauli [2] into (*).

$$(\gamma_\mu\Pi_\mu-m)\Psi=0. \qquad (1)$$

Despite of simple appearance it is a system of 4 equations for 4-component bispinor Ψ.

Evidently $\hat{\Pi}^2\Psi=m^2\Psi$. A little exercise in algebra gives

$$\hat{\Pi}^2=\left(E^2+2\frac{3E}{r}+\frac{3^2}{r^2}\right)+\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r-\frac{\vec{L}^2}{r^2}\right)-i3\gamma_0\vec{\gamma}\frac{\vec{n}}{r^2}. \qquad (2)$$

Notations $k^2=E^2-m^2, \hat{O}=\vec{L}^2-3^2-i3\gamma_0(\vec{\gamma}\vec{n})$ transform the equation into the form

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r+k^2+2\frac{3E}{r}-\frac{\delta}{r^2}\right)\Psi=0. \qquad (3)$$

It is seen that radial and angular variables are separated. Taking $\Psi=\frac{R(r)}{r}\Phi(\vec{n})$ we divide the equation into two ones: radial (with $\rho=-2ikr; v=3E/r$) and angular (with $\hat{O}$)

$$\left(\frac{\partial^2}{\partial\rho^2}-\frac{1}{4}+\frac{iv}{\rho}-\frac{\lambda}{\rho^2}\right)\rho R(\rho)=0 \text{ and } \hat{O}\Phi=\lambda\Phi. \qquad (4)$$

Both equations have two solutions. Radial functions $R(\rho)$ are $\mathcal{M}$ and $\mathcal{W}$ solutions of Whittaker equation (3),

$$R(\rho)=\frac{a\mathcal{M}_{iv,\mu}(\rho)+b\mathcal{W}_{iv,\mu}(\rho)}{\rho}, (\mu^2=\lambda+1), \qquad (5)$$

having identical indices and arguments but different asymptotic behavior There are also two angular functions $\Phi_\pm$, corresponding to states with $j=l\pm1/2$: $\Phi=\alpha\Phi_+ +\beta\Phi_-$.

Thus, we have obtained four linear independent solutions forming a complete system. The specific choice of them depends on energy E and angular momentum j: for $E < m$ the $\mathcal{M}$ must be taken (and for $E \geq m$ — the $\mathcal{W}$). A similar rule applies to angular functions: Φ_+ for $j=l+1/2$ and Φ_- for $j=l-1/2$.

II. Analysis

Being solution of 2-nd order equation did Ψ satisfy the 1-st order equation? The answer is yes, but along with $(\hat{\Pi}-m)\Psi=0$ it satisfies also $(\hat{\Pi}+m)\Psi=0$. We may return $+m$ back to $-m$ changing γ_μ into $\gamma_5\gamma_\mu\gamma_5$ by means of transformation $\Psi\to\gamma_5\Psi$ which is equivalent to reversing all signs in Π's. So, this class of solutions has e > 0 and is going backwards in space — it is a positron!

The next question is symmetry of found solutions. Defining the "*Whittaker Hamiltonian*" $\mathcal{H}=\rho\partial^2-\frac{\rho}{4}-\frac{\lambda}{\rho}$ the *Whittaker* equation becomes $\mathcal{H}\Psi=iv\Psi$. Along with $S_1=\mathcal{H}$ there exist $S_2=\rho\partial^2+\frac{\rho}{4}-\frac{\lambda}{\rho}$, commuting with S_1 according to $[S_1, S_2] = iS_3$, where $S_3 = i\rho\partial$. The algebra closes with $[S_2, S_3] = iS_1$ and $[S_3, S_1] = -iS_2$.

This algebra is called $SO(2,1)$ [3]. It includes the "ladder" operator $S_+ = S_2 + iS_3$ which commutes with Hamiltonian as $\mathcal{H}S_+ - S_+\mathcal{H}=iS_+$. Acting on Ψ_v we have $\mathcal{H}(S_+\Psi_v)-ivS_+\Psi_v=iS_+\Psi_v$. It means that $S_+\Psi_v=a\Psi_{v+1}$ — operator S_+ realizes movement along the spectrum of ν with unit step! In other words, $v = v_0+N$, where N = 0,1,2…

* Notations: $\hat{\Pi}=\gamma_\mu\Pi_\mu=\gamma_0(E-eA_0)-\vec{\gamma}(\vec{p}-e\vec{A})$; In Coulomb field $eA_0=-3/r, \vec{A}=0, 3=Ze^2$; As usual $\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\delta_{\alpha\beta}$. Bispinor Ψ has four components depending on r, ϑ, φ.

Thus, the symmetry quantizes Coulomb parameter ν and associated energy $E=\frac{m}{\sqrt{1+3^2\nu^{-2}}}$.*

III. Remarks

Why *Dirac* had written $(\vec{\alpha}\vec{p}+\beta m)\Psi=E\Psi$? The answer is evident: to obtain the connection between energy and momentum $\vec{p}^2+m^2=E^2$ (it follows from $\vec{\alpha}\beta=-\beta\vec{\alpha}$).

However, choice $\beta = \rho_3$, $\vec{\alpha}=\rho_1\vec{\sigma}$ (adopted by *Dirac* and followers) is not the most successful, especially for proving the relativistic invariance. Of course, this is not a mistake but flaw.

It was corrected by *Pauli* [2] — in his hands the equation obtained relativistic and gauge-invariances which are almost obvious.

As for me, the choice $\gamma_0 = \rho_1$, $\vec{\gamma}=i\rho_2\vec{\sigma}$ seems more convenient**, in particular spin — orbit inter-action $-i3\gamma_0(\vec{\gamma}\vec{n})r^{-2}$ becomes diagonal $\rho_3(\vec{\sigma}\vec{n})3/r^2$. Nevertheless, the *Dirac* choice is still used by almost everyone (excluding *V.Fock*). I hope that Part I is convincing enough to demonstrate the benefits of my choice.

Another example — an interpretation of "low" spinor χ as positron wave function – is simply wrong. In reality positron wave function $\Psi^{(p)}=\gamma_5\Psi^{(e)}$ (see Part II) needs **both** components φ and χ.

These remarks are addressed to my colleagues, performing complicated calculations of physical processes – their complexity is often connected with unsuccessful choice of one or another representation. For example, momentum representation is very comfortable in scattering events but completely failed in bound states.

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* Its relation to *Sommerfeld* formula is discussed in [4].
** As is shown in "The Lamb shift" (to be published)