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МАТЕМАТИКА В ТАБЛИЦАХ. ЧАСТЬ II.

Учебно-методическое пособие.

ВЕЛИКИЙ НОВГОРОД
2015

Математика в таблицах содержит материалы на английском языке к занятиям по основам математического анализа, одного из главных и трудных разделов высшей математики. Пособие может быть полезно как преподавателям, читающим соответствующий курс лекций, так и студентам, изучающим основы математического анализа (темы – производные высших порядков, приложения производной, неопределенный интеграл). Использование этих материалов позволит сделать преподавание основ указанного раздела более доступным и наглядным. Поскольку таблицы представлены на английском языке, то они могут быть использованы преподавателями, обучающими студентов-математиков английскому языку, а также их учениками. Пособие предназначено прежде всего для преподавателей и студентов высших учебных заведений, хотя отдельные его части можно применять в работе с учениками в классах с углубленным изучением математики.

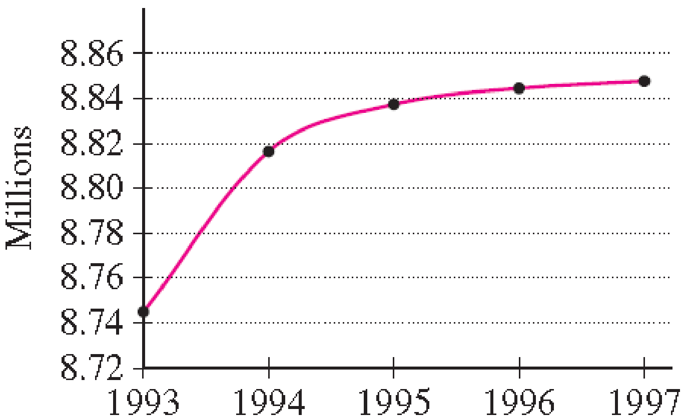
2.6. Higher-order derivatives

Higher derivatives are the functions obtained by repeatedly differentiating a function $y = f(x)$.

If f' is differentiable, **the second derivative**, denoted f'' (read "*f double prime*"), is the derivative of f':

$$\boxed{f''(x) = \frac{d}{dx}\left(f'(x)\right)}$$

The second derivative is the rate of change of $f'(x)$



Population $P(t)$ of Sweden (in millions).The rate of increase declined in the period 1993–1997.

Population of Sweden

Year	1993	1994	1995	1996	1997
Population	8,745,109	8,816,381	8,837,496	8,844,499	8,847,625
Yearly increase		71,272	21,115	7,003	3,126

- The population $P(t)$ of Sweden has increased every year for more than 100 years (therefore, the first derivative $P'(t)$ is positive).
- Table shows that the rate of yearly increase declined dramatically in the years 1993–1997.
- So although $P'(t)$ was still positive in these years, $P'(t)$ decreased and therefore the second derivative $P''(t)$ was negative in the period 1993–1997.

2.6. Higher-order derivatives

Derivative	Prime Notation	Operator Notation	Leibniz Notation
First	y'	$D_x y$	$\frac{dy}{dx}$
Second	y''	$D_x^2 y$	$\frac{d^2y}{dx^2}$
Third	y'''	$D_x^3 y$	$\frac{d^3y}{dx^3}$
Fourth	$y^{(4)}$	$D_x^4 y$	$\frac{d^4y}{dx^4}$
:	:	:	:
*n*th	$y^{(n)}$	$D_x^n y$	$\frac{d^ny}{dx^n}$

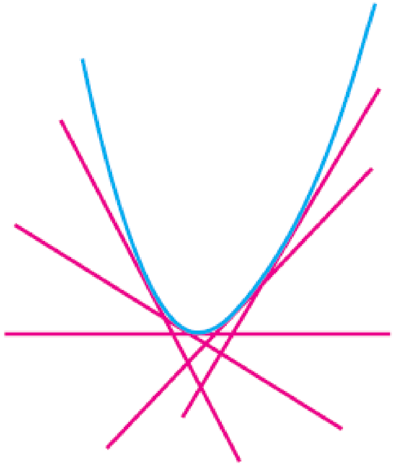
Leibniz's notation for the second derivative is read *the second derivative of y with respect to x*

$\frac{dy}{dx}$ has units of y per unit of x;

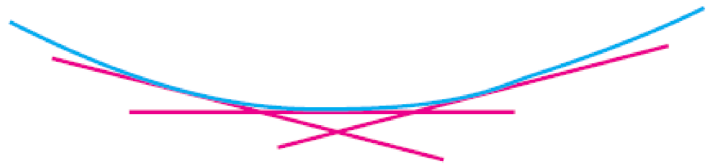
$\frac{d^2y}{dx^2}$ has units of $\frac{dy}{dx}$ per unit of x, or units of y per unit of x-squared.

2.6. Higher-order derivatives

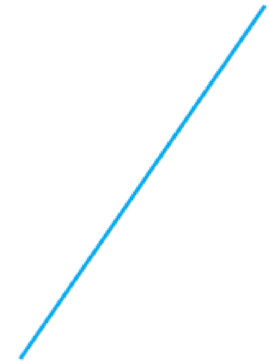
Can we visualize the rate represented by a second derivative?



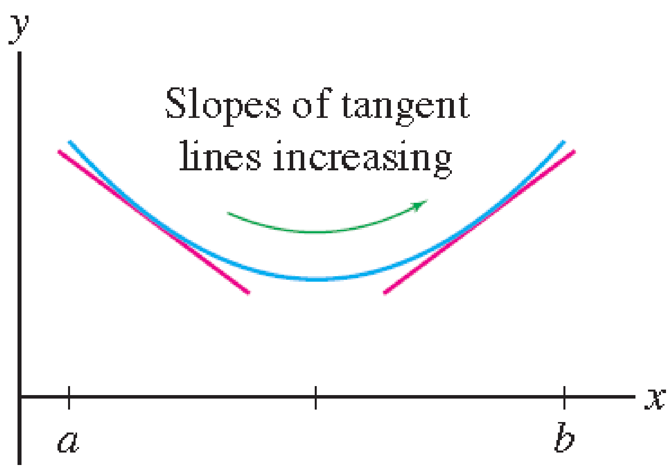
(A) Large second derivative:
Tangent lines turn rapidly.



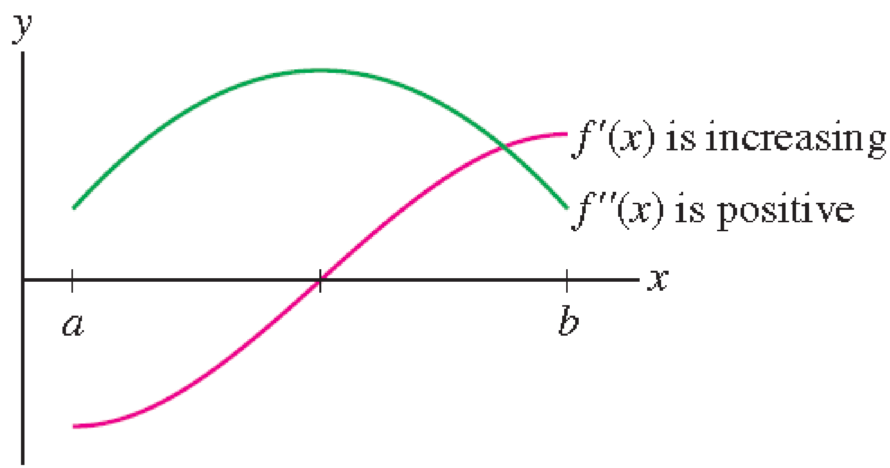
(B) Smaller second derivative:
Tangent lines turn slowly.



(C) Second derivative is zero:
Tangent line does not change.



Graph of $f(x)$



Graph of first two derivatives

2.6. Higher-order derivatives

If $s = s\,(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the ***instantaneous velocity*** of the object as a function of time:

$$\boxed{v(t) = s'(t) = \frac{ds}{dt} = \dot{s}(t)}$$

The instantaneous rate of change of velocity with respect to time is called the ***acceleration*** of the object:

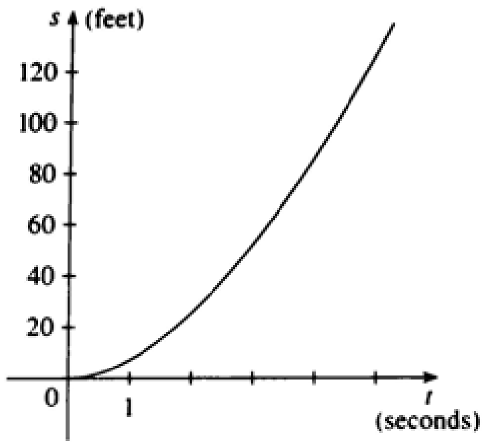
$$\boxed{\begin{aligned} a(t) &= v'(t) = s''(t) \\ a(t) &= \frac{dv}{dt} = \frac{d^2s}{dt^2} = \ddot{s}(t) \end{aligned}}$$

The third derivative of the position function is the derivative of the acceleration function and is called the ***jerk***:

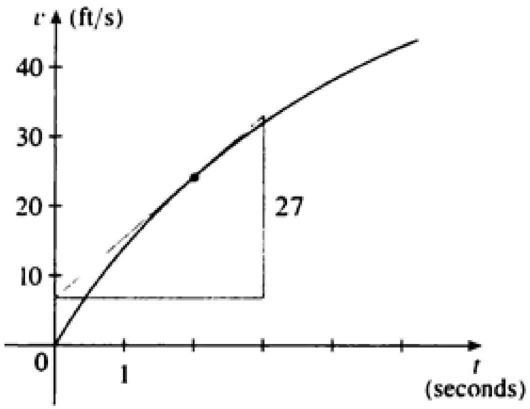
$$\boxed{j = \frac{da}{dt} = \frac{d^3s}{dt^3} = \dddot{s}(t)}$$

The jerk is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

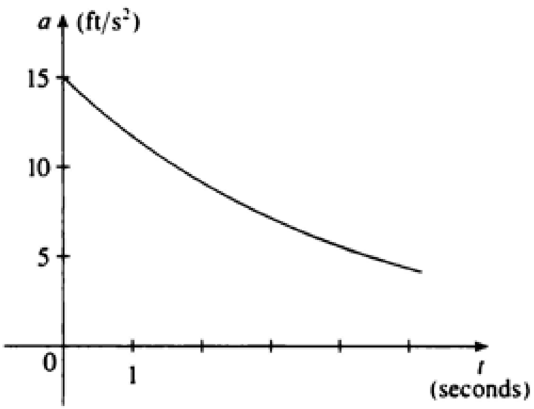
2.6. Higher-order derivatives



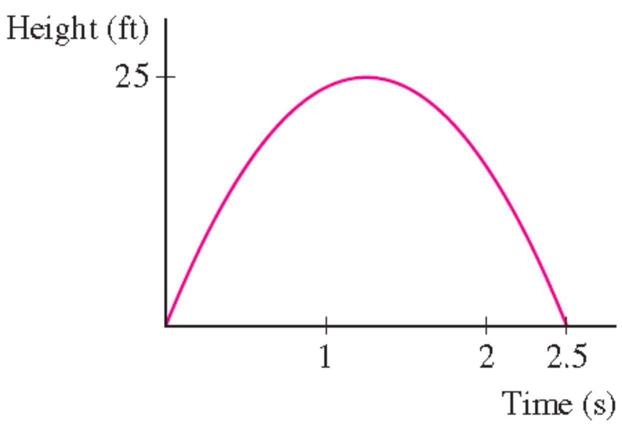
Position function

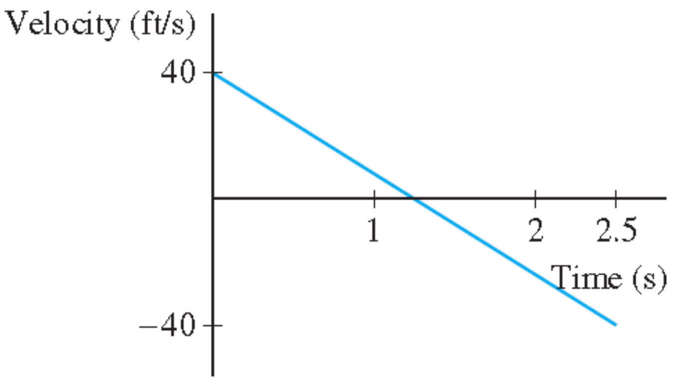


Velocity function



Acceleration function



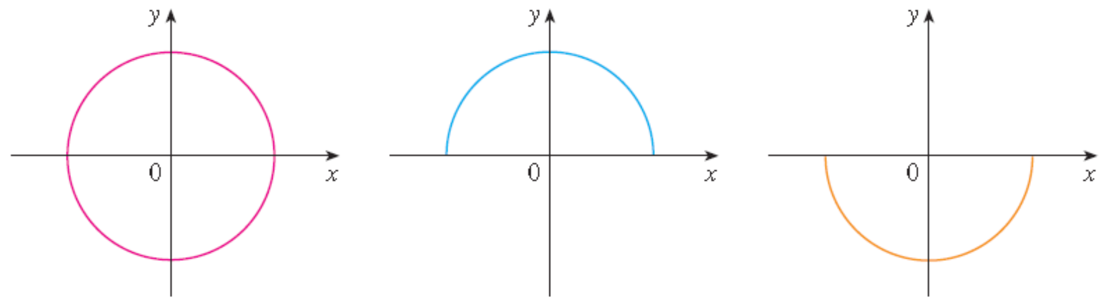


Height and velocity functions

2.7. Implicit differentiation

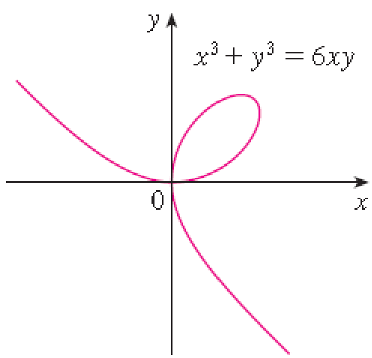
The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable, in general, $y = f(x)$. Some functions, however, are defined ***implicitly*** by a relation between x and y.

$$x^2 + y^2 = 25$$

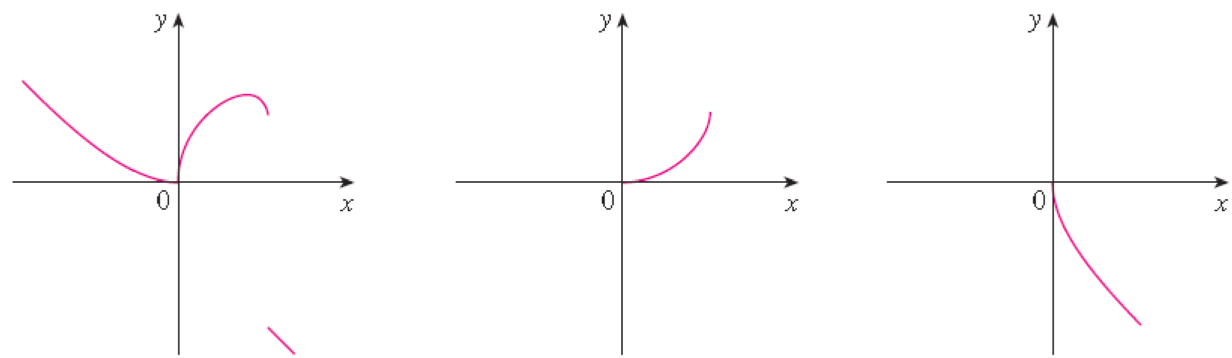


(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

When we say that is a function defined implicitly by equation $x^3 + y^3 = 6xy$, we mean that the equation $x^3 + [f(x)]^3 = 6xf(x)$ is true for all values of x in the domain of f.



The folium of Descartes



Graphs of three functions defined by the folium of Descartes

2.7. Implicit differentiation

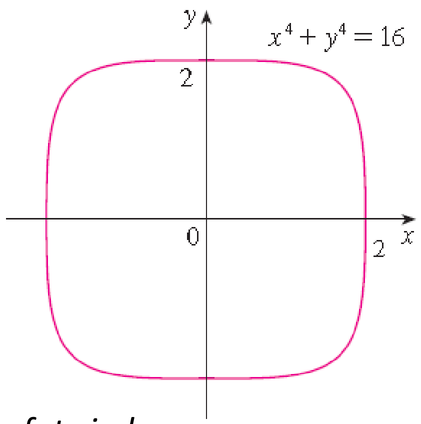
- We don't need to solve an equation for y in terms of x in order to find the derivative of y. We can use the method of **implicit differentiation**.
- This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.
- Here it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Algorithm:

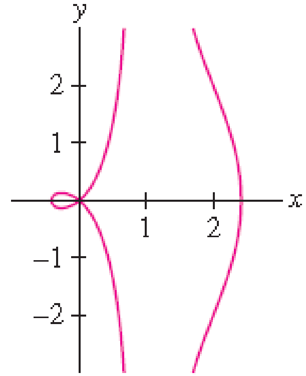
- Implicit differentiation is used to compute dy/dx when x and y are related by an equation.
- *Step 1.* Take the derivative of both sides of the equation with respect to x.
- *Step 2.* Solve for y' by collecting the terms involving y' on one side and the remaining terms on the other side of the equation.
- Remember to include the factor dy/dx when differentiating expressions involving y with respect to x. For instance,

$$\frac{d}{dx}\sin y = (\cos y)\frac{dy}{dx}$$

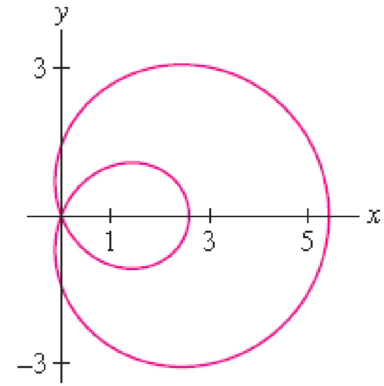
2.7. Implicit differentiation



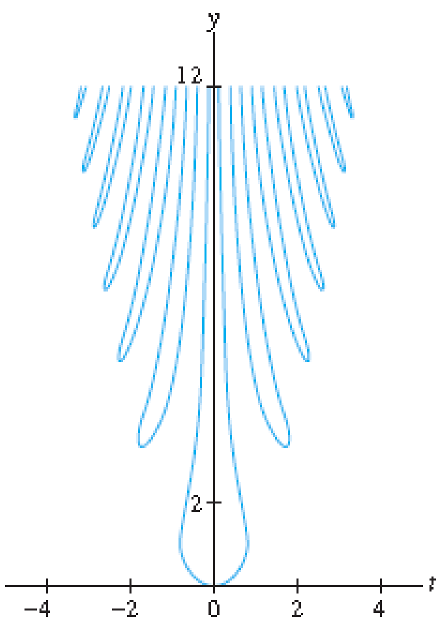
a fat circle



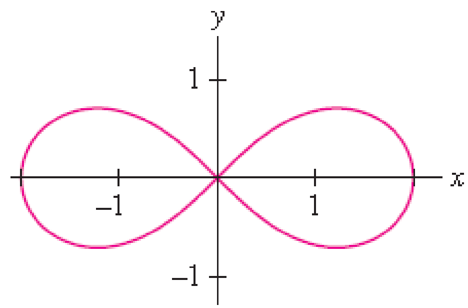
Conchoid: $(x-1)^2(x^2+y^2) = 2x^2$.



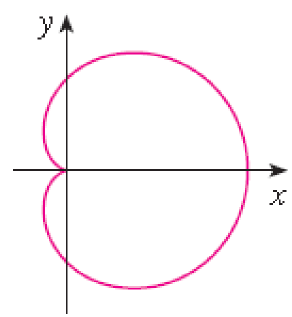
Limaçon: $(x^2+y^2-4x)^2 = 2(x^2+y^2)$.



Graph of $\cos(ty) = \frac{t^2}{y}$.

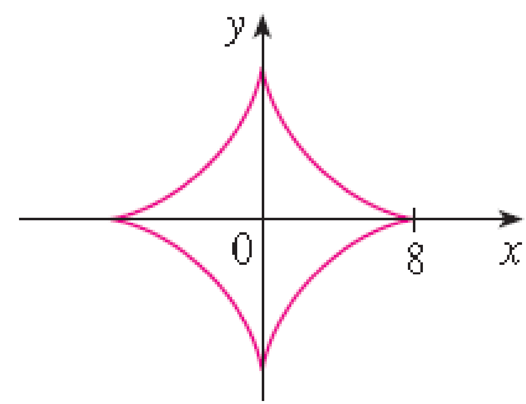


Lemniscate curve: $(x^2+y^2)^2 = 4(x^2-y^2)$.



cardioid

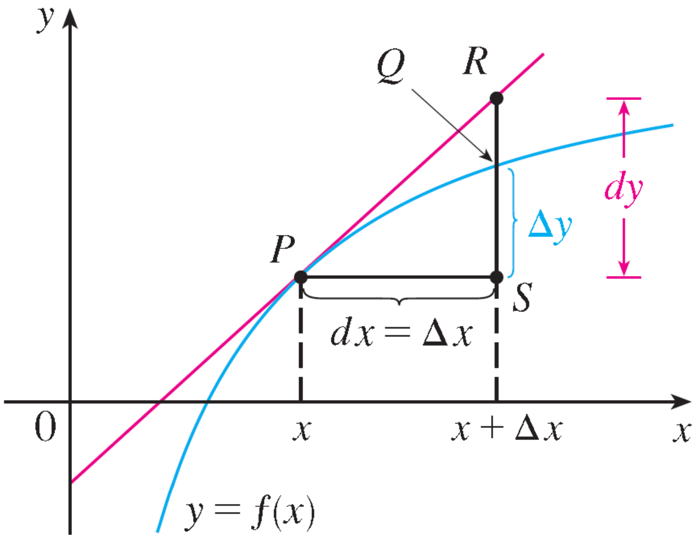
$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$



$$x^{2/3} + y^{2/3} = 4$$

astroid

2.9. Differentials and approximations



- $P\,(x, f\,(x))$, $Q\,(x+\Delta x, f\,(x+\Delta x))$
- $dx = \Delta x,\ \Delta y = f\,(x+\Delta x) - f\,(x)$
- the slope of the tangent line PR is the derivative $f\,'(x)$
- the directed distance from S to R is $f\,'(x)dx = dy$.
- dy represents the amount that the tangent line rises or falls; Δy represents the amount that the curve rises or falls when x changes by an amount dx.

Def. Differentials

- Let $y = f\,(x)$ be a differentiable function of the independent variable x.
- Δx is an arbitrary increment in the independent variable x.
- dx, called the **differential of the independent variable** x, is equal to Δx.
- Δy is the actual change in the variable y as x changes from x to $x + \Delta x$; that is, $\Delta y = f\,(x+\Delta x) - f\,(x)$.
- dy, called the **differential of the independent variable** y, is defined by

$$\boxed{dy = f\,'(x)dx}$$

2.9. Differentials and approximations

Differential Rule

$$dk = 0, \quad k = const \qquad d(ku) = k\,du, \quad k = const$$

$$d(u+v) = du + dv \qquad d(uv) = u\,dv + v\,du$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \qquad d(u^n) = nu^{n-1}du$$

Derivatives and differentials are not the same!!!!

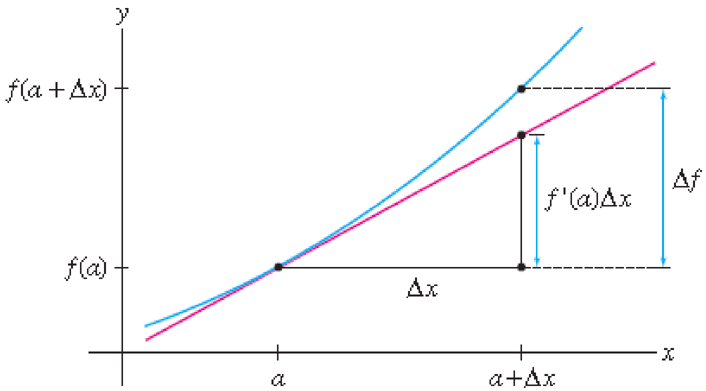
$\frac{dy}{dx}$ - a symbol for the derivative

dy - a symbol for the differential

2.9. Differentials and approximations

Suppose that $y = f(x)$. An increment Δx produces a corresponding increment Δy in y, which can be approximated by dy. Thus, $f(x + \Delta x)$ is approximated by

$$\boxed{f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\Delta x}$$



Linear Approximation of Δf

If f is differentiable at $x = a$ and Δx is small, then

$$\boxed{\Delta f \approx f'(a)\Delta x}$$

where $\Delta f = f(a+\Delta x) - f(a)$.

Approximating $f(x)$ by Its Linearization Assume that f is differentiable at $x = a$.

If x is close to a, then

$$\boxed{f(x) \approx L(x) = f'(a)(x-a) + f(a)}$$

The error in the Linear Approximation is the quantity $\text{Error} = \left|\Delta f - f'(a)\Delta x\right|$

In many cases, *the percentage error* is more relevant than the error itself:

$$\text{Percentage error} = \left|\frac{\text{error}}{\text{actual value}}\right| \times 100\%$$

2.8. Related rates

- ***Related rate problems*** present us with situations in which one or more variables are related by an equation.
- In related rate problems, the goal is to calculate an unknown rate of change in terms of other rates of change that are known. This will usually require implicit differentiation.

Algorithm:

Draw a diagram if possible.

- ***Step 1.* Assign variables and restate the problem.**
- ***Step 2.* Find an equation that relates the variables and differentiate.**

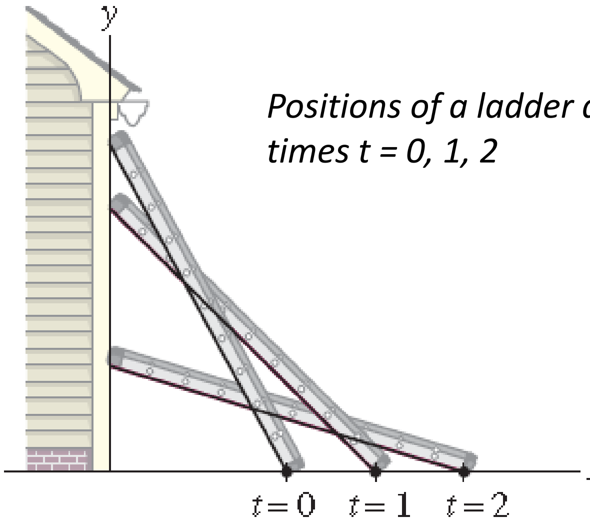
This gives us an equation relating the known and unknown derivatives. Remember not to substitute values for the variables until after you have computed all derivatives.

- ***Step 3.* Use the given data to find the unknown derivative.**

The two facts from geometry that arise most often in related rate problems are the Pythagorean Theorem and the Theorem of Similar Triangles (ratios of corresponding sides are equal).

2.8. Related rates

How fast does the top of the ladder move if the bottom of the ladder is pulled away from the wall at constant speed?



Ladder Problem. A 16-ft ladder leans against a wall. The bottom of the ladder is 5 ft from the wall at time $t = 0$ and slides away from the wall at a rate of 3 ft/s. Find the velocity of the top of the ladder at time $t = 1$.

Solution. ***Step 1. Assign variables and restate the problem.***

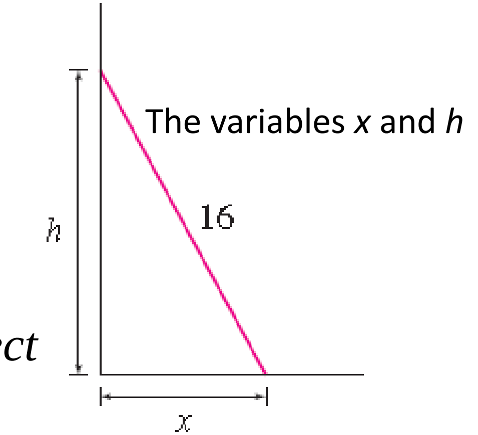
Since we are considering how the top and bottom of the ladder change position, we use variables:

- $x = x(t)$ distance from the bottom of the ladder to the wall
- $h = h(t)$ height of the ladder's top

Both x and h are functions of time. The velocity of the bottom is $dx/dt = 3$ ft/s. Since the velocity of the top is dh/dt and the initial distance from the bottom to the wall is $x(0) = 5$, we can restate the problem as

$$\boxed{\text{Compute } \frac{dh}{dt} \text{ at } t = 1 \text{ given that } \frac{dx}{dt} = 3 \text{ ft/s and } x(0) = 5\text{ft}}$$

2.8. Related rates



Step 2. Find an equation that relates the variables and differentiate.

To solve this problem, we need an equation relating x and h. This is provided by the Pythagorean Theorem: $x^2 + h^2 = 16$

To calculate dh/dt, we differentiate both sides of this equation *with respect to t*: $\frac{d}{dt}x^2 + \frac{d}{dt}h^2 = \frac{d}{dt}16$ $\quad 2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

This yields $\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$ and since $\frac{dx}{dt} = 3$ ft/s, the velocity of the top is

$$\boxed{\frac{dh}{dt} = -3\frac{x}{h} \text{ ft/s}}$$

Step 3. Use the data to find the unknown derivative.

To apply this formula, we must find x and h at time $t = 1$.

Since the bottom slides away at 3 ft/s and $x(0) = 5$,

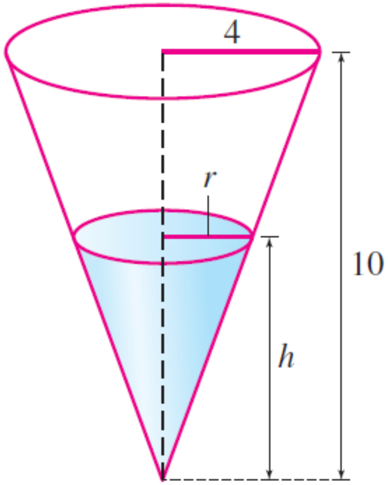
we have $x(1) = 8$ and $h(1) = \sqrt{16^2 - 8^2} \approx 13.86$

$$\left.\frac{dh}{dt}\right|_{t=1} = -3\frac{x(1)}{h(1)} \approx -1.7 \text{ ft/s}$$

t	x	h	dh/dt
0	5	15.20	−0.99
1	8	13.86	−1.73
2	11	11.62	−2.84
3	14	7.75	−5.42

This table of values confirms that the top of the ladder is speeding up.

2.8. Related rates



Filling a Conical Tank. Water pours into a conical tank of height 10 ft and radius 4 ft at a rate of 10 ft^3/min. How fast is the water level rising when it is 5 ft high?

Step 1. Assign variables and restate the problem.

Let V and h be the volume and height of the water in the tank at time t . The problem is

$$\boxed{\text{Compute } \frac{dh}{dt} \text{ at } h = 5 \text{ given that } \frac{dV}{dt} = 10 \text{ ft}^3/\text{min}}$$

Step 2. Find an equation that relates the variables and differentiate.

The volume is $V = \frac{1}{3}\pi h r^2$, where r is the radius of the cone at height h, *but we cannot use this relation unless we eliminate the variable r*. Using similar triangles, we see that $\frac{r}{h} = \frac{4}{10}$; $r = 0.4h$

$$V = \frac{1}{3}\pi h(0.4h)^2 = \frac{1}{3}\pi(0.16)h^3 \quad \frac{dV}{dt} = (0.16)\pi h^2 \frac{dh}{dt}$$

Do not set h = 5 until the end of the problem, after the derivatives have been computed. This applies to all related rate problems.

Step 3. Use the data to find the unknown derivative.

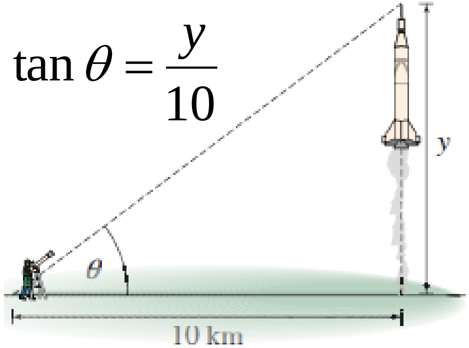
Using the given data $dV/dt = 10$, we have $(0.16)\pi h^2 \frac{dh}{dt} = 10$

$$\boxed{\frac{dh}{dt} = \frac{10}{(0.16)\pi h^2} \approx \frac{20}{h^2}}$$

When $h = 5$, the level is rising at $dh/dt \approx 20/5^2 = 0.8$ ft^3/min.

2.8. Related rates

$$\tan\theta = \frac{y}{10}$$



Tracking a Rocket. A spy tracks a rocket through a telescope to determine its velocity. The rocket is traveling vertically from a launching pad located 10 km away. At a certain moment, the spy's instruments show that the angle between the telescope and the ground is equal to π/3 and is changing at a rate of 0.5 rad/min. What is the rocket's velocity at that moment?

Step 1. Assign variables and restate the problem.

Let θ be the angle between the telescope and the ground, and let y be the height of the rocket at time t. Then our goal is to compute the rocket's velocity dy/dt when $\theta = \pi/3$. We restate the problem as follows:

$$\boxed{\text{Compute } \left.\frac{dy}{dt}\right|_{\theta=\frac{\pi}{3}} \text{ given that } \frac{d\theta}{dt} = 0.5 \text{ rad/min when } \theta = \pi/3}$$

Step 2. Find an equation that relates the variables and differentiate.

We need a relation between θ and y. Now differentiate with respect to time:

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}; \quad \frac{dy}{dt} = \frac{10}{\cos^2\theta}\frac{d\theta}{dt}$$

Step 3. Use the given data to find the unknown derivative.

$$\boxed{\frac{dy}{dt} = \frac{10}{\cos^2(\pi/3)}(0.5) = \frac{5}{\cos^2(\pi/3)}}$$

At the given moment, $\theta = \pi/3$ and $d\theta/dt = 0.5$.

$$\frac{dy}{dt} = \frac{5}{\cos^2(\pi/3)} = \frac{10}{(0.5)^2}(0.5) = 20 \text{ km/min}$$

The rocket's velocity at this moment is 20 km/min or 1,200 km/hour.

The Derivative

$$\underbrace{f'(x)}_{\text{Lagrange's notation}} = \underbrace{\frac{dy}{dx}}_{\text{Leibniz's notation}} = \underbrace{D_x f(x)}_{\text{Euler's notation}} = \underbrace{\dot{f}(x)}_{\text{Newton's notation}} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The simple functions

$$f'(x)=0,\quad f(x)=k=const$$

$$f'(x)=1,\quad f(x)=x$$

$$(kf)'=k\cdot f',\quad k=const$$

$$f'(x^n)=nx^{n-1}$$

$$(\sin x)'=\cos x \qquad (\cos x)'=-\sin x$$

$$(\tan x)'=\sec^2 x \qquad (\cot x)'=-\csc^2 x$$

$$(\sec x)'=\sec x\tan x$$

$$(\csc x)'=-\csc x\cot x$$

Differentiation rules

$$(f+g)'=f'+g'$$

$$(f-g)'=f'-g'$$

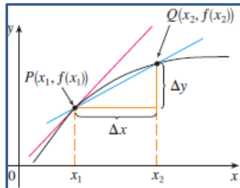
$$(f\cdot g)'=fg'+gf'$$

$$\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$$

The composite functions

$$\boxed{(f(g(x))'=f'(g(x))\cdot g'(x)}$$

The derivative of a composite function is the derivative of the outer function evaluated at the inner function, times the derivative of the inner function



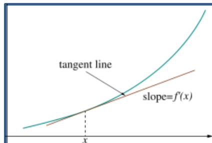
If this limit does exist, we say that f is differentiable at x.

If the derivative $f'(c)$ exists at the point c, then f is continuous function at point c.

*The **function f '** can be evaluated at any point; To evaluate it at a particular point, we write something like:* $\left.\frac{dy}{dx}\right|_{x=x_0}$

$D_x=\frac{d}{dx}$ *is a differential operator.*

The tangent line



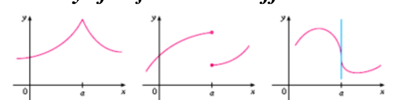
$$\boxed{m_{sl}=f'(x_0)}$$

The derivative at a point of a function is the slope of the tangent line to the graph of the function at that point.

The equation of the tangent line at point x_0:

$$y-y_0=m_{sl}(x-x_0)$$

Three ways for f not to be differentiable at c



A corner *A discontinuity* *A vertical tangent*

An acceleration

$v(t)=s'(t)=\frac{ds}{dt}$ *- the first derivative of the position function represents the **instantaneous velocity** of the object as a function of time*

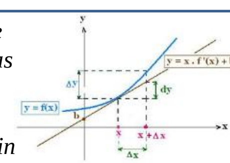
$a(t)=v'(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}=s''(t)$ *- the instantaneous rate of change of velocity with respect to time is called the **acceleration***

$s(t)=s_0+v_0t-16t^2$ *- the formula of the position function (in feet) for an object which is thrown straight upward (or downward) in the gravity field.*

The Differential

If the derivative of function f (x) at the point x is f '(x), we define the differential of the function f (x) by df (x) such as

$$\boxed{dy=f'(x)dx}$$



The differential dx represents an infinitely small change in the variable x.

$$dk=0,\quad k=const \qquad d(u+v)=du+dv \qquad d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

$$d(ku)=kdu,\quad k=const \qquad d(uv)=udv+vdu \qquad d(u^n)=nu^{n-1}du$$

Implicit differentiation

Implicit differentiation is used to compute *dy/dx* when *x* and *y* are related by an equation.

Step 1. Take the derivative of both sides of the equation with respect to *x*.

Step 2. Solve for *y′* by collecting the terms involving *y′* on one side and the remaining terms on the other side of the equation.

Remember to include the factor *dy/dx* when differentiating expressions involving *y* with respect to *x*.

Related rate problems

The goal is to calculate an unknown rate of change in terms of other rates of change that are known. This will usually require implicit differentiation.

Algorithm:

Step 1. Assign variables and restate the problem.

Step 2. Find an equation that relates the variables and differentiate.

Step 3. Use the given data to find the unknown derivative.

Linear approximation

Assume that *f* is differentiable at *x* = *a*. If *x* is close to *a*, then

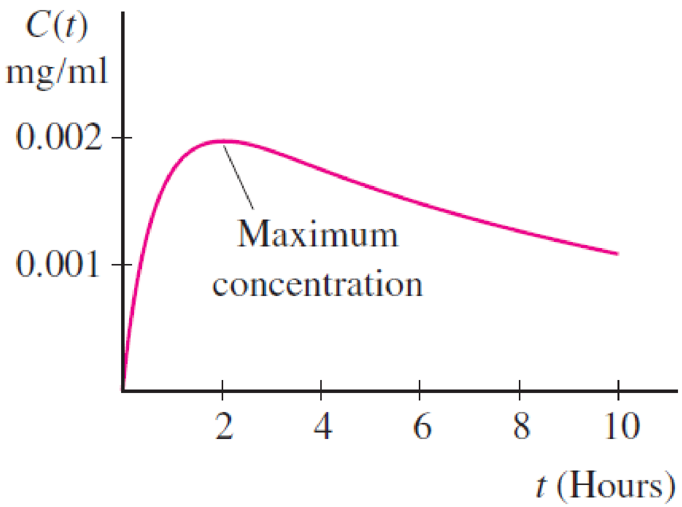
$$L(x)=f'(a)(x-a)+f(a)$$

is the *linearization* of *f* at *x* = *a*. The Linear Approximation can be rewritten as the estimate $f(x)\approx L(x)$ for small $|x-a|$.

$$f(x+\Delta x)\approx f(x)+dy=f(x)+f'(x)\Delta x$$

- a good approximation for finding roots and powers of the numbers.

3.1. Maxima and Minima



$C(t)$ = *drug concentration in bloodstream*

- A physician must determine the maximum drug concentration in a patient's bloodstream when a drug is administered.
- This amounts to finding the highest point on the graph of $C(t)$, the concentration at time t.

Some of the most important applications of differential calculus are *optimization problems,* in which we are required to find the optimal (best) way of doing something:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration).
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what price should a business sell its products in order to maximize revenue?

3.1. Maxima and Minima

Def. Maxima and minima

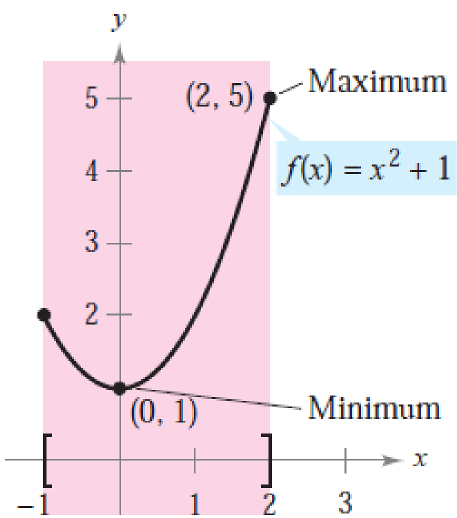
Let S, the domain of f, contain the point c. We say that

1. $f(c)$ is the **maximum value** of f on S if $f(c) \geq f(x)$ for all x in S;

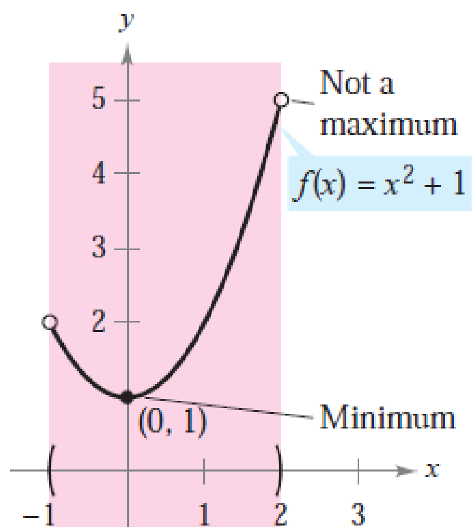
2. $f(c)$ is the **minimum value** of f on S if $f(c) \leq f(x)$ for all x in S;

3. $f(c)$ is an **extreme value (extremum)** of f on S if it is either the maximum value or the minimum value;

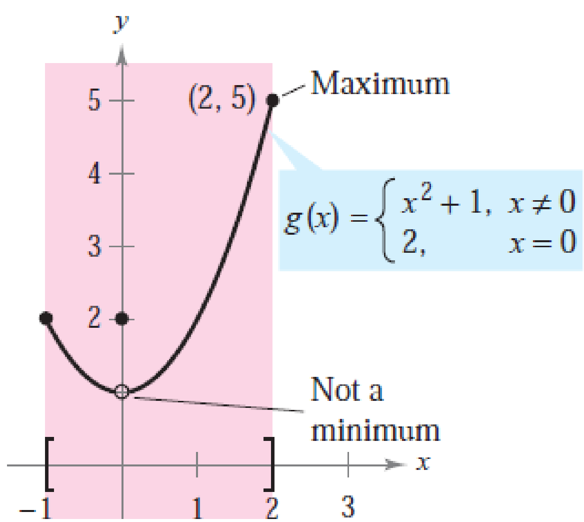
4. the function we want to maximize or minimize is the **objective function**.



f is continuous, [-1, 2] is closed.



f is continuous, (-1, 2) is open.

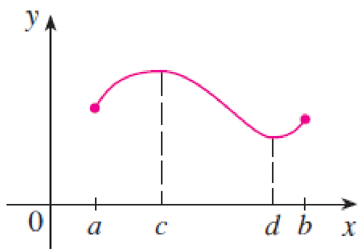


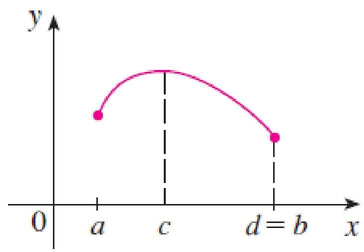
g is not continuous, [-1, 2] is closed.

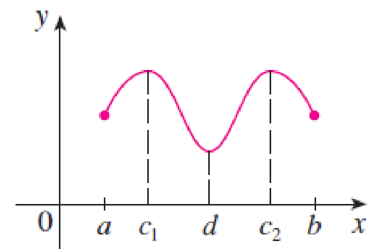
3.1. Maxima and Minima

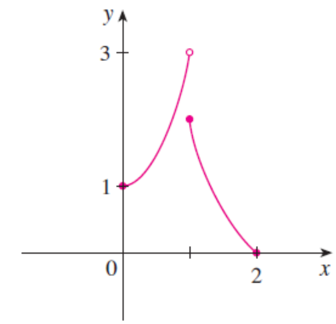
Theorem A. Max-Min Existence Theorem (Extreme Value Theorem)

If f is **continuous** on a **closed** interval $[a,b]$, then f attains both a maximum value and a minimum value there.

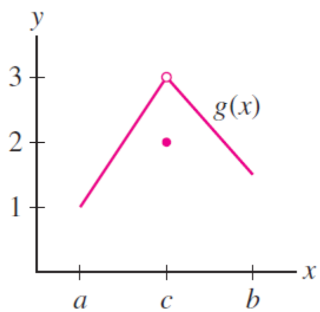




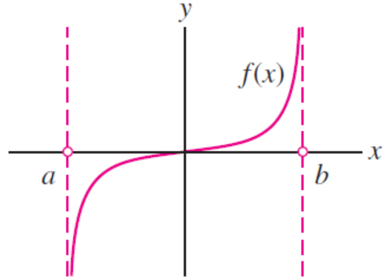




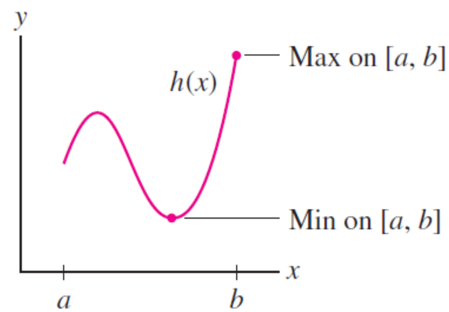
This function has minimum value $f(2)=0$, but no maximum value.



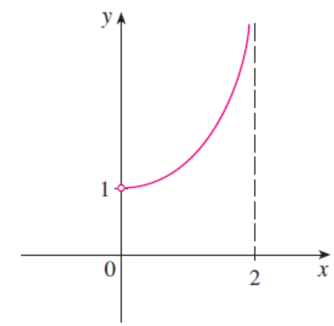
$g(x)$ is discontinuous and has no max on $[a, b]$.



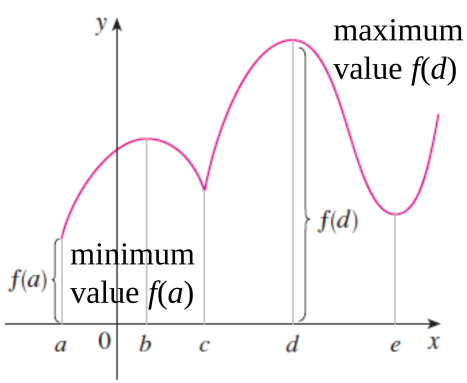
$f(x)$ has no min or max on the open interval (a, b).

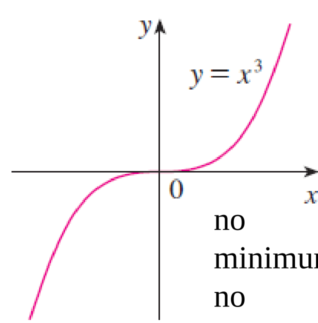


$h(x)$ is continuous and $[a, b]$ is closed. Therefore, $h(x)$ has a min and max on $[a, b]$.

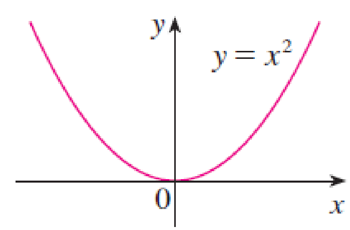


This continuous function g has no maximum or minimum.



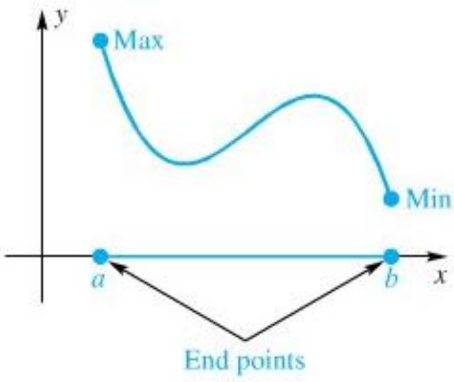


no minimum, no maximum

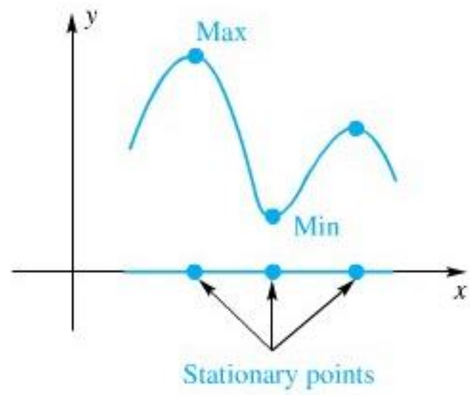


Minimum value 0, no maximum

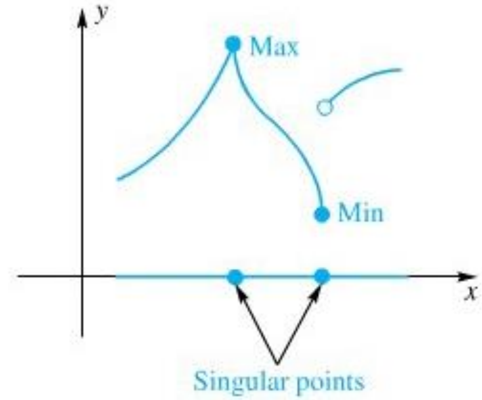
3.1. Maxima and Minima



Extrema can occur at **endpoints** of an interval.



If c is a point at which $f'(c) = 0$, we call c a **stationary point**.



If c is an interior point of an interval where f' fails to exist, we call c a **singular point**.

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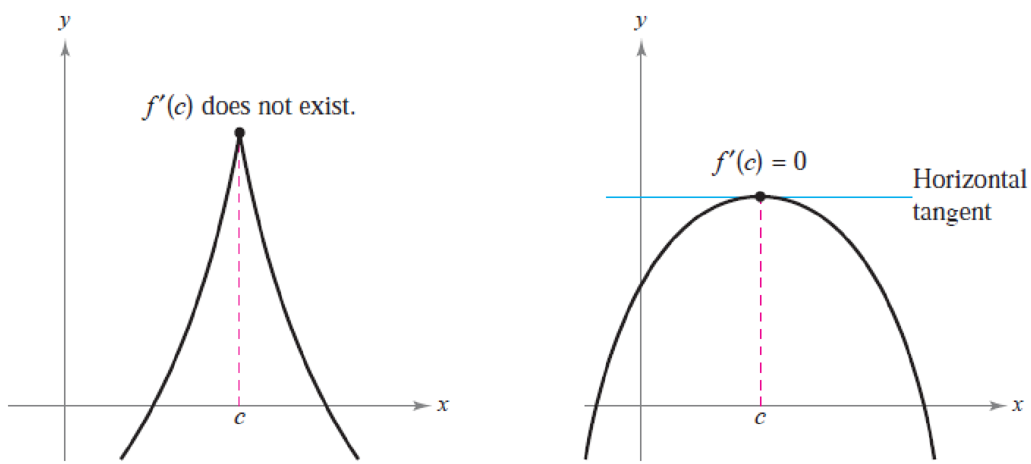
Def. Critical point

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical point of f.

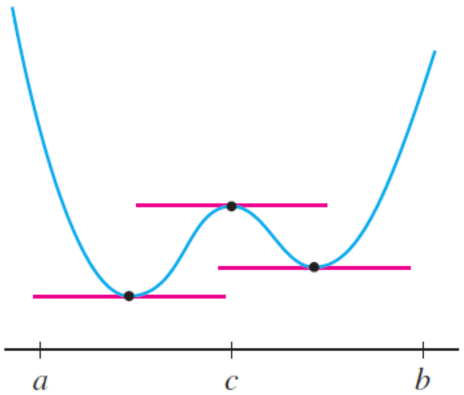


c is a critical point of f

3.1. Maxima and Minima

Theorem B. Critical Point Theorem (Fermat's Theorem)

Let f be defined on an interval I containing the point c. **If $f(c)$ is an extreme value, then c must be a critical point**; that is, either c is an end point of I or a stationary point of f or a singular point of f.



GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical points of f in (a, b).

2. Evaluate f at each critical point in (a, b).

3. Evaluate f at each endpoint of $[a, b]$.

4. The least of these values is the minimum. The greatest is the maximum.

3.2. Monotonicity and Concavity

Def. Increasing / Decreasing behavior of functions

Let f be defined on an interval I (open, closed, or neither). We say that

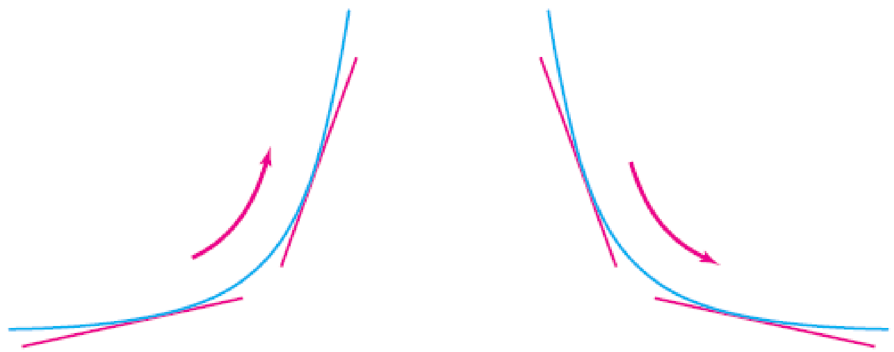
f is **increasing** on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

f is **decreasing** on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

f is **strictly monotonic** on I if it is either increasing on I or decreasing on I.



Increasing function
Tangent lines have positive slope.

Decreasing function
Tangent lines have negative slope.

3.2. Monotonicity and Concavity

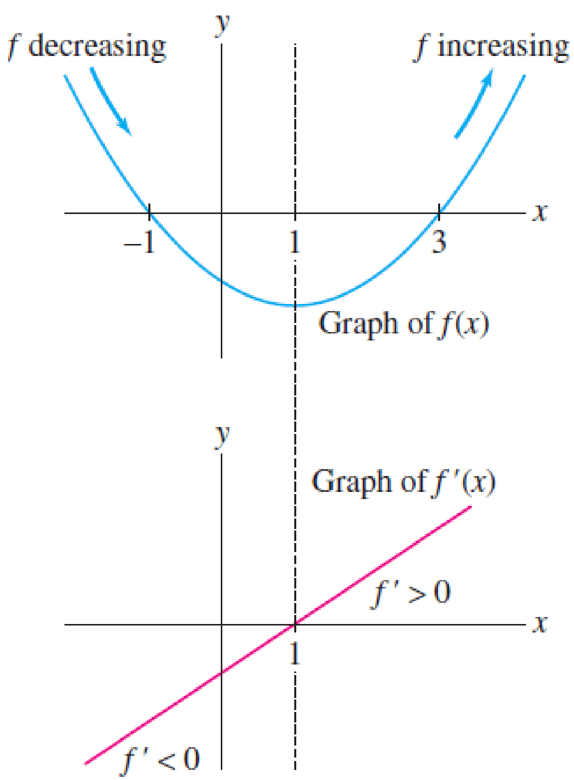
Theorem A. Monotonicity Theorem

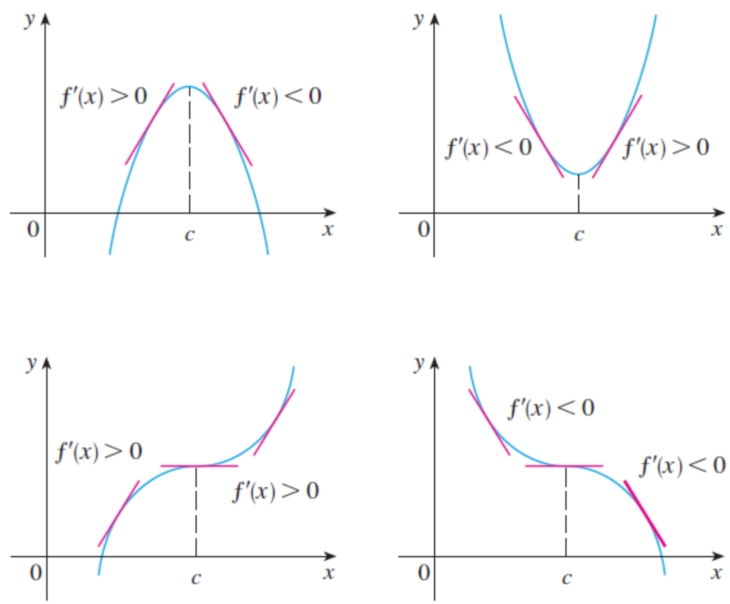
Let f be *continuous* on an interval I and *differentiable* at every interior point of I.

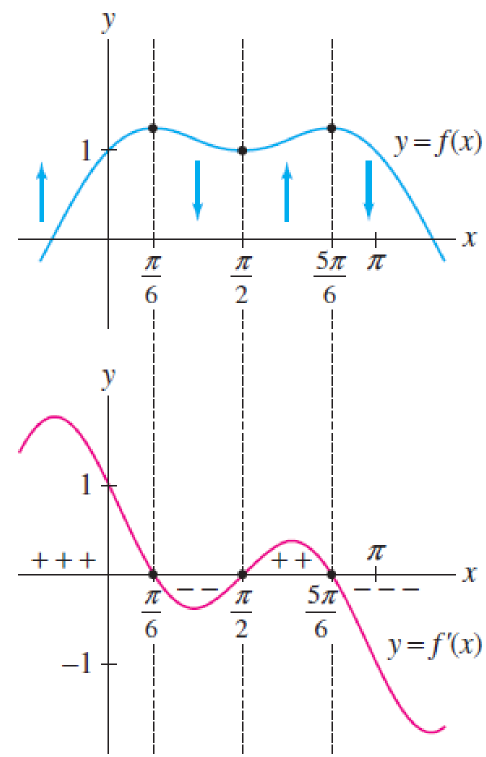
We say that

If $f'(x) > 0$ for all x interior to I, then f is **increasing** on I.

If $f'(x) < 0$ for all x interior to I, then f is **decreasing** on I.







3.2. Monotonicity and Concavity

Def. Concavity

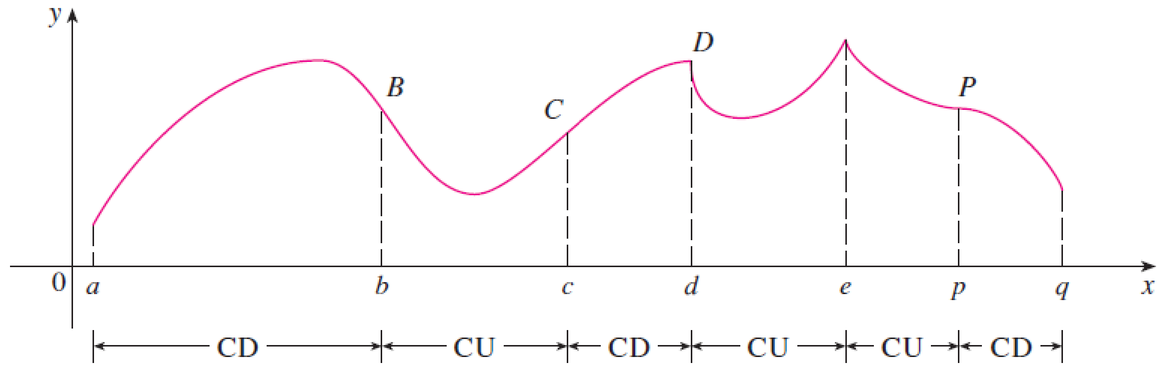
Let f be differentiable on an open interval I. We say that

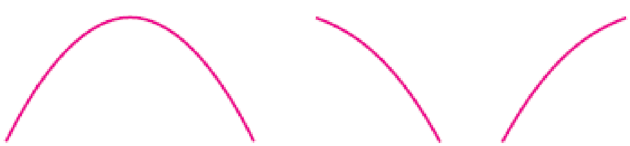
f (as well as its graph) is **concave up** on I if f' is *increasing* on I.

f (as well as its graph) is **concave down** on I if f' is *decreasing* on I.

Concave up

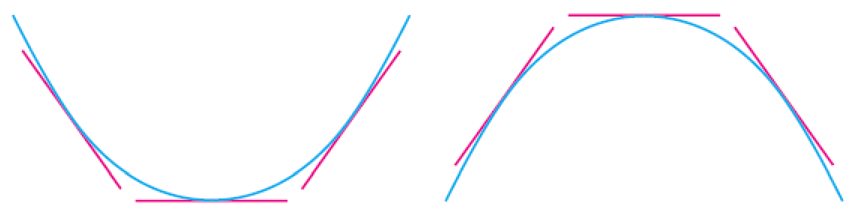
a curve is called *concave up* if it bends up





Concave down

a curve is called *concave down* if it bends down



Concave up:
Slopes of tangent lines
are increasing.

Concave down:
Slopes of tangent lines
are decreasing.

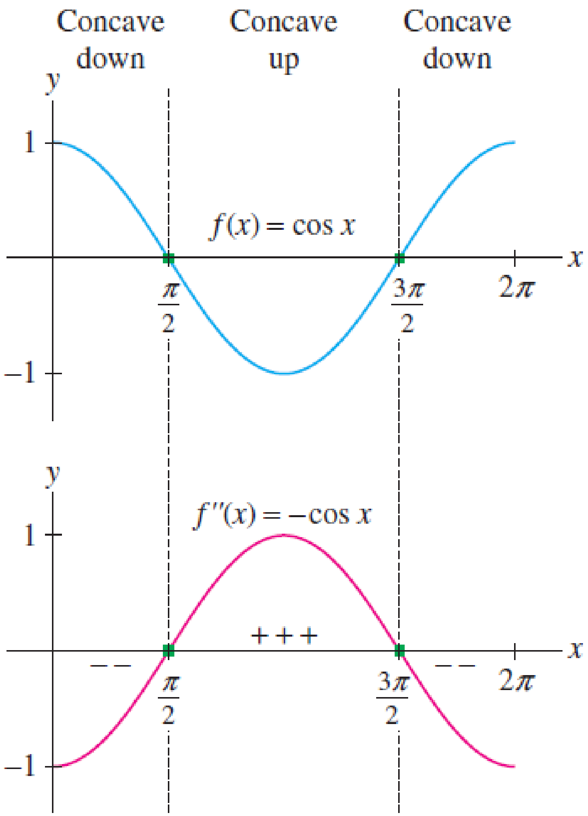
3.2. Monotonicity and Concavity

Theorem B. Concavity Theorem

Let f be *twice differentiable* on the open interval I.

If $f''(x) > 0$ for all x in I, then f is **concave up** on I.

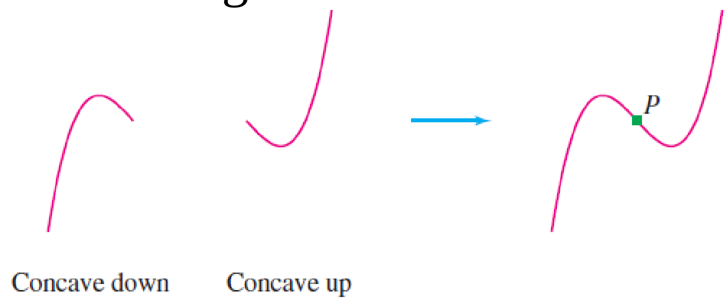
If $f''(x) < 0$ for all x in I, then f is **concave down** on I.

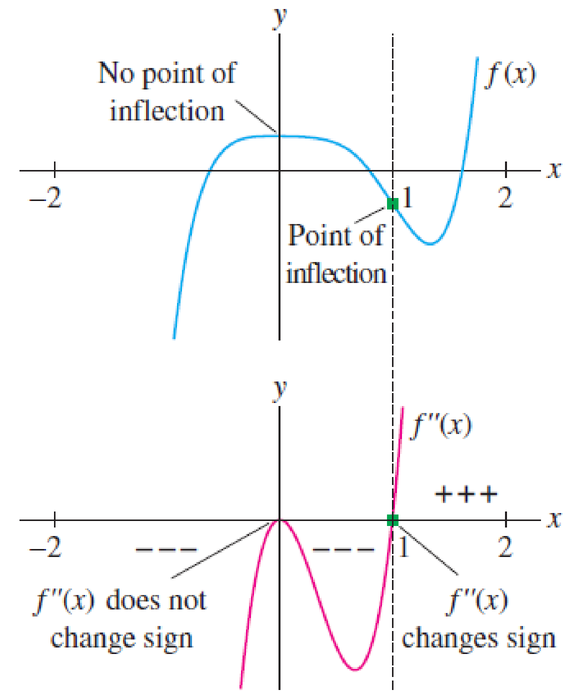


First Derivative	Second Derivative
$f' > 0 \Rightarrow f$ is increasing	$f'' > 0 \Rightarrow f$ is concave up
$f' < 0 \Rightarrow f$ is decreasing	$f'' < 0 \Rightarrow f$ is concave down

3.2. Monotonicity and Concavity

We say that P = (c, $f(c)$) is a **point of inflection** of $f(x)$ if the concavity changes from up to down or vice versa at $x = c$. In other words, $f'(x)$ is increasing on one side of $x = c$ and decreasing on the other.

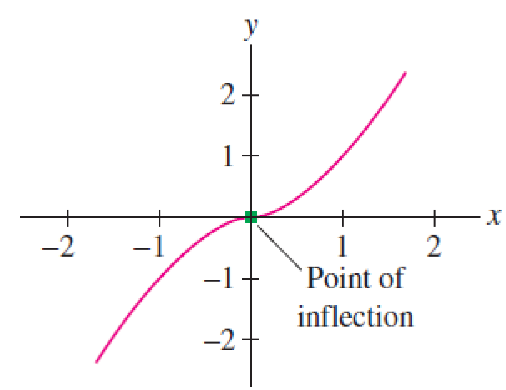




Theorem. Test for Inflection Points

If $f''(c) = 0$ and $f''(x)$ changes sign at c, then c is a **point of inflection**.

Points where $f''(x) = 0$ or where $f''(x)$ does not exist are the *candidate* for the points of inflection.



3.3. Local and Global Extrema

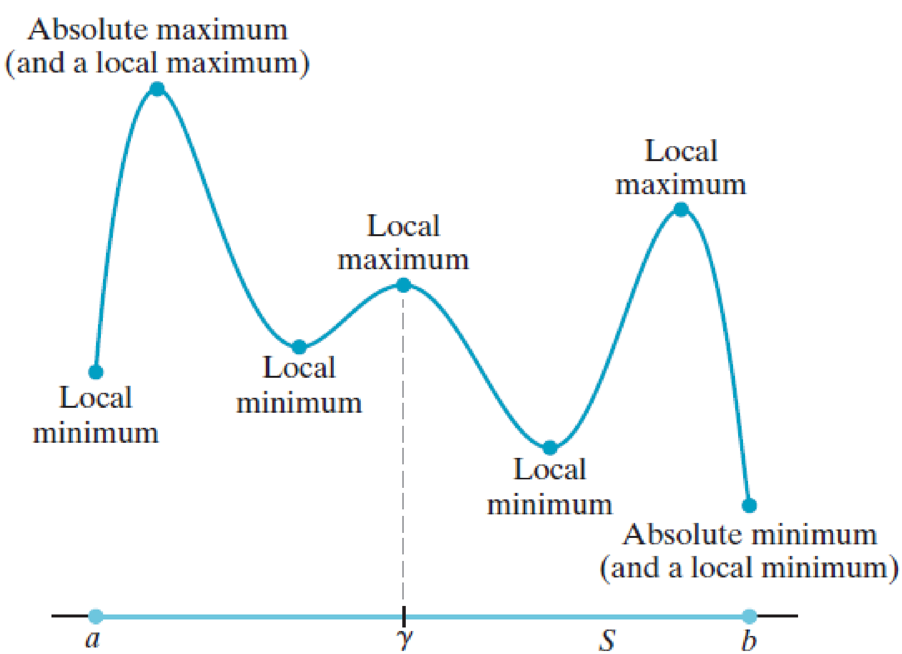
Def. Maxima and minima

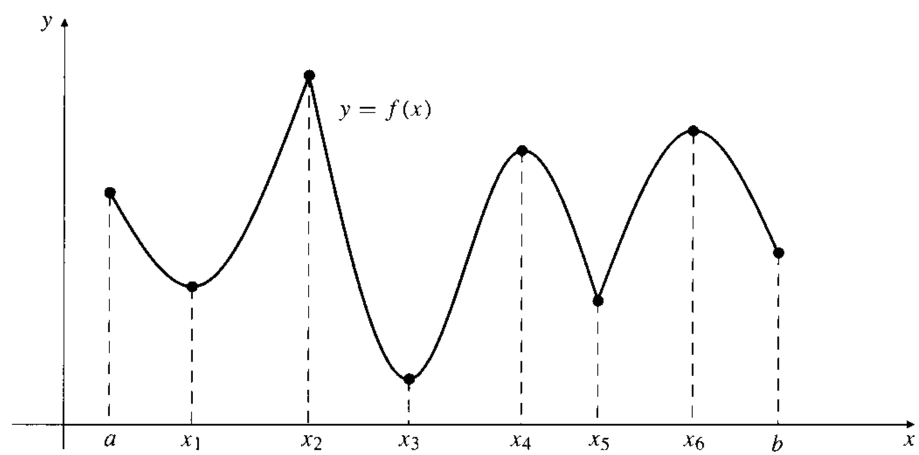
Let S, the domain of f, contain the point c. We say that

1. $f(c)$ is the **global (absolute) maximum** of f on S if $f(c) \geq f(x)$ for all x in S;

2. $f(c)$ is the **global (absolute) minimum** of f on S if $f(c) \leq f(x)$ for all x in S;

3. $f(c)$ is an **global (absolute) extremum** of f on S if it is either the maximum value or the minimum value.





The absolute maximum is the highest of the local maxima;
the absolute minimum is the lowest of the local minima.

3.3. Local and Global Extrema

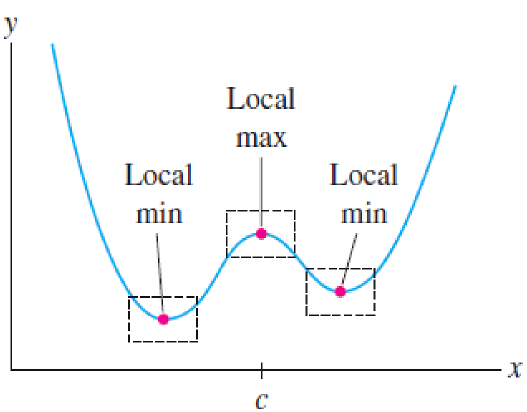
Def. Maxima and minima

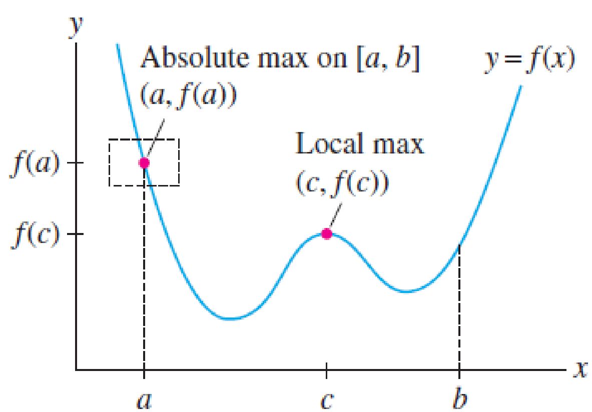
Let S, the domain of f, contain the point c. We say that

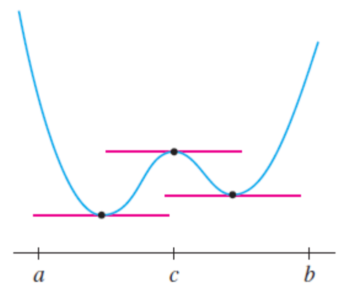
1. $f(c)$ is a **local (relative) maximum** of f if there is an interval (a, b) containing c such that $f(c)$ is the maximum value of f on $(a, b) \cap S$; (the symbol $\cap$ denotes the intersection (common part) of two sets)

2. $f(c)$ is a **local (relative) minimum** of f if there is an interval (a, b) containing c such that $f(c)$ is the minimum value of f on $(a, b) \cap S$;

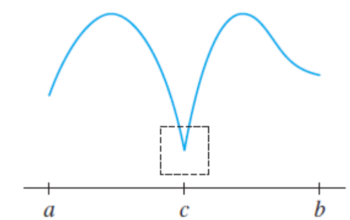
3. $f(c)$ is a **local (relative) extremum** of f if it is either a local maximum value or a local minimum value.







Tangent line is horizontal at the local extrema.



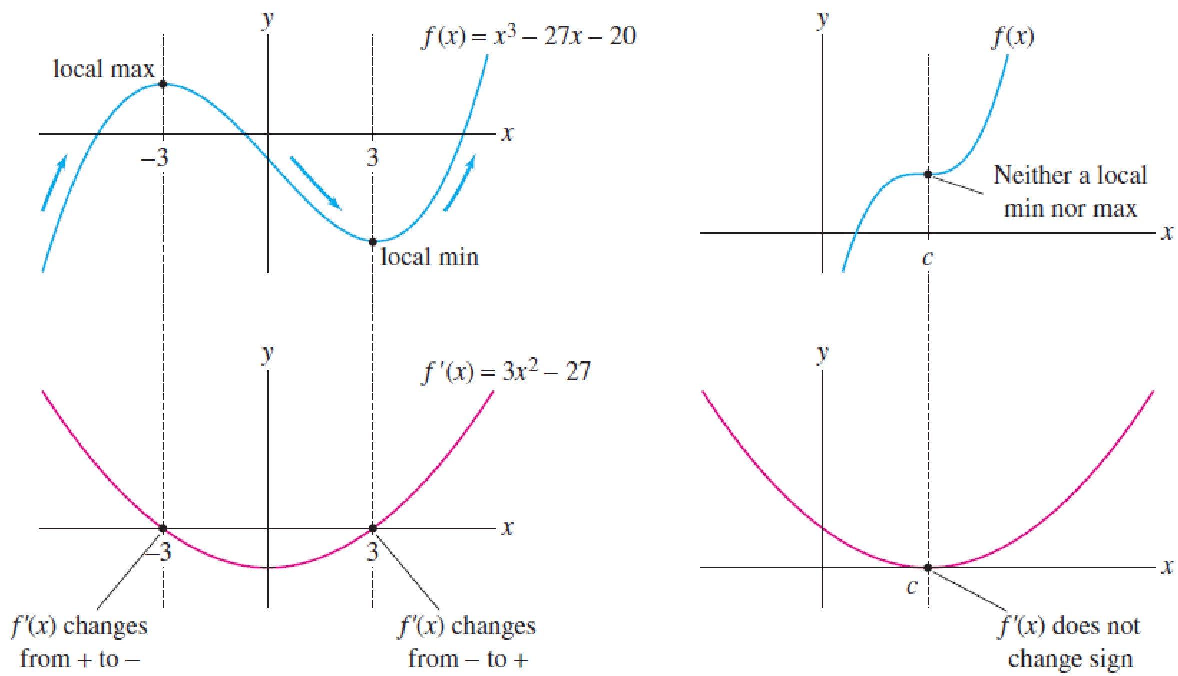
This local minimum occurs at a point where the function is not differentiable.

3.3. Local and Global Extrema

Theorem A. First Derivative Test

Let f be continuous on an open interval (a, b) that contains a critical point c.

1. If $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b), then $f(c)$ is a local maximum of f.
2. If $f'(x) < 0$ for all x in (a, c) and $f'(x) > 0$ for all x in (c, b), then $f(c)$ is a local minimum of f.
3. If $f'(x)$ has the same sign on both sides of c, then $f(c)$ is not a local extreme value of f.



Sign Change of f' at c	Type of Critical Point
From + to –	Local maximum
From – to +	Local minimum

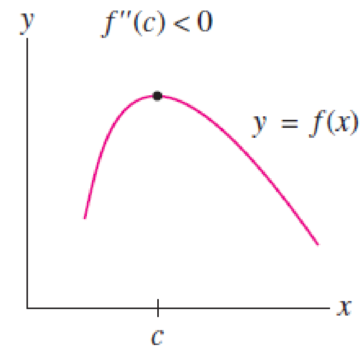
If f' does not change sign at c, then f has no local maximum or minimum at c.

3.3. Local and Global Extrema

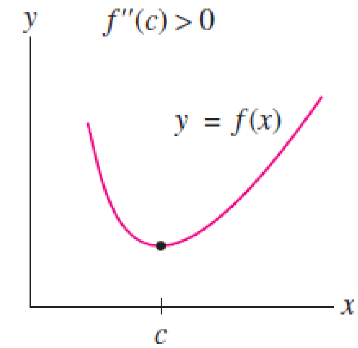
Theorem B. Second Derivative Test

Let $f'(x)$ and $f''(x)$ exist at every point in an open interval (a, b) containing c, and suppose that $f'(c) = 0$.

1. If $f''(c) < 0$, then $f(c)$ is a local maximum of f.
2. If $f''(c) > 0$, then $f(c)$ is a local minimum of f.

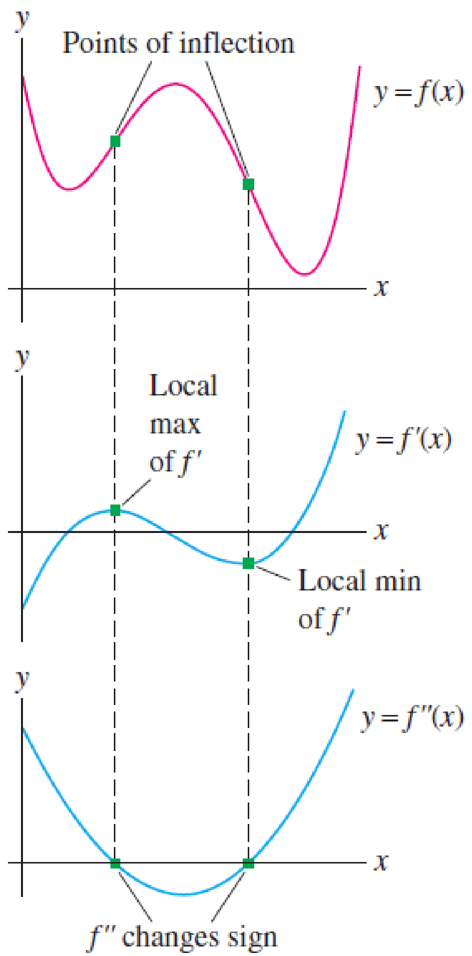


Concave down—local max

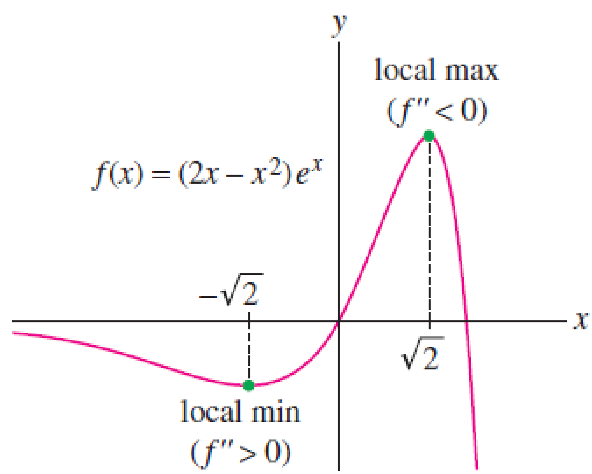


Concave up—local min

Concavity determines whether the critical point is a local minimum or maximum.



Inflection points of f occur where $f'(x)$ has a local min or max



- $f(c)$ is a local maximum if $f''(c) < 0$.
- $f(c)$ is a local minimum if $f''(c) > 0$.
- The test fails if $f''(c) = 0$.

3.4. Optimization Problems

Some of the most important applications of differential calculus are *optimization problems,* in which we are required to find the optimal (best) way of doing something.

STEPS IN SOLVING OPTIMIZATION PROBLEMS

1. Understand the Problem The first step is to read the problem carefully until it is clearly understood. Ask yourself: *What is the unknown? What are the given quantities? What are the given conditions?*

2. Draw a Diagram In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

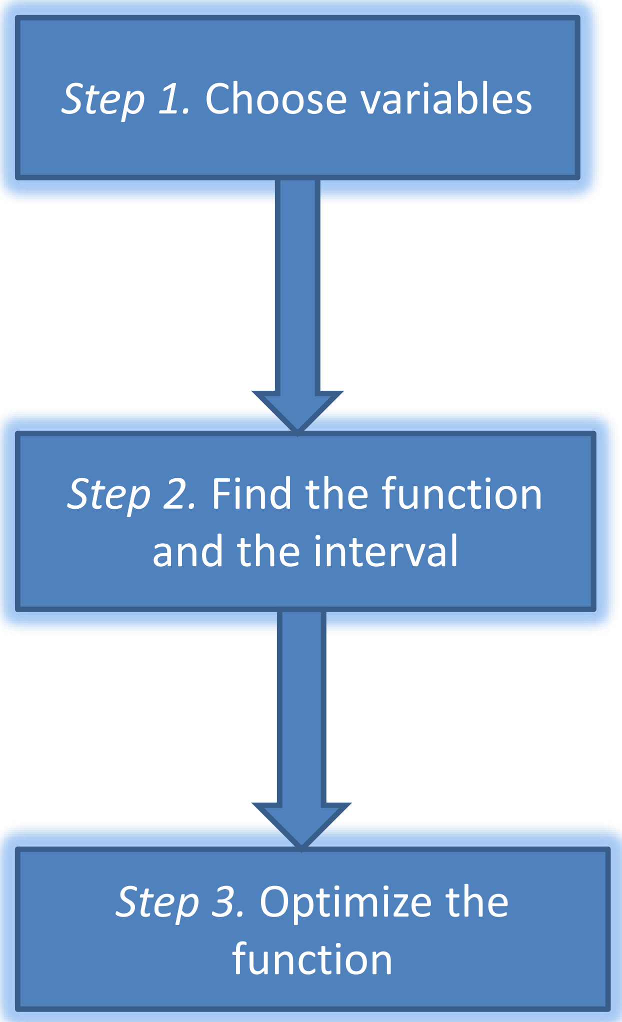
3. Introduce Notation Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function.

6. Use the theorems (the first and the second derivatives tests) to find the maximum or minimum value of f.

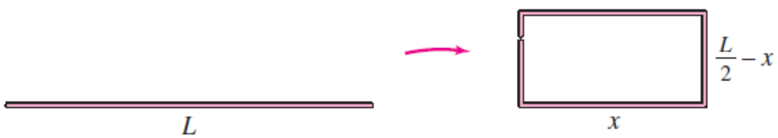
3.4. Optimization Problems



3.4. Optimization Problems

1. Geometrical example (maximizing area)

A piece of wire of length L is bent into the shape of a rectangle. Which dimensions produce the rectangle of maximum area?



Step 1. Choose variables

If the rectangle has sides of length x and y, then its area is $A = xy$.

Since A depends on two variables x and y, we cannot find the maximum until we eliminate one of the variables.

The perimeter of the rectangle is L and $2x + 2y = L$ $\rightarrow$ we can get $y = L/2 - x$

An equation relating two or more variables in an optimization problem is called a "constraint equation."

Step 2. Find the function and the interval

$$\boxed{\begin{array}{l} A(x) = x\left(\dfrac{L}{2} - x\right) = \left(\dfrac{L}{2}\right)x - x^2 \\ 0 \le x \le L/2 \end{array}}$$

Over which interval does the optimization take place?

$x \ge 0$

$L/2 - x \ge 0$

the sides of the rectangle cannot have negative length

Step 3. Optimize the function

Our problem reduces to finding the maximum of A (x) on the closed interval [0, L/2].

Solving $A'(x) = L/2 - 2x = 0$, we find that $x = L/4$ is the critical point.

End points: $A(0) = 0$ Critical point:

$$A\left(\frac{L}{2}\right) = \frac{L}{2}\left(\frac{L}{2} - \frac{L}{2}\right) = 0 \qquad A\left(\frac{L}{4}\right) = \frac{L}{4}\left(\frac{L}{2} - \frac{L}{4}\right) = \frac{L^2}{16}$$

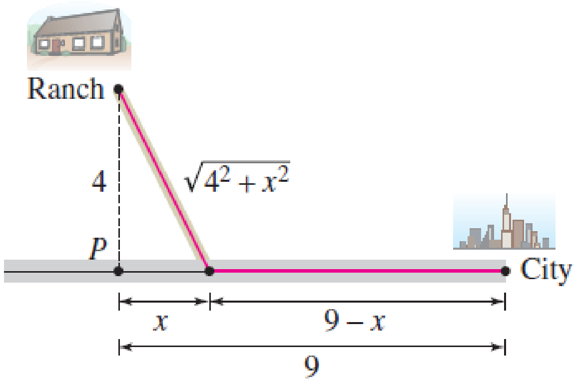
The largest area occurs for $x = L/4$ and $y = \dfrac{L}{2} - x = \dfrac{L}{2} - \dfrac{L}{4} = \dfrac{L}{4}$

The rectangle of maximum area is the square of sides $x = y = L/4$.

3.4. Optimization Problems

2. Physical example (minimizing time)

Cowboy Clint wants to build a dirt road from his ranch to the highway so that he can drive to the city in the shortest amount of time. The perpendicular distance from the ranch to the highway is 4 miles, and the city is located 9 miles down the highway. Where should Clint join the dirt road to the highway if the speed limit is 20 mph on the dirt road and 55 mph on the highway?



Step 1. Choose variables

We need to decide where the dirt road should join the highway.

Let x be the distance from P (the point on the highway nearest the ranch) to the point where the dirt road joins the highway.

We need to compute the travel time $T(x)$ of the trip as a function of x.

By the Pythagorean Theorem, the length of the dirt road is $\sqrt{4^2+x^2}$

The time required to travel a distance d at constant velocity v is $t = d/v$.

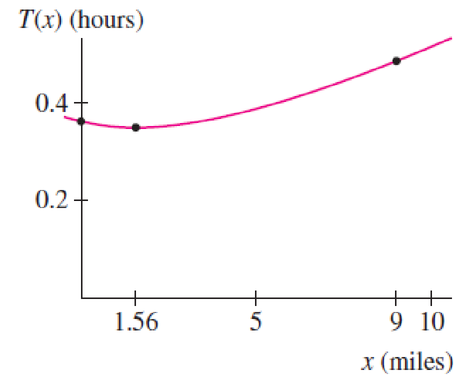
Applying this with $v = 20$ mph

$\dfrac{\sqrt{4^2+x^2}}{20}$ hours to traverse the dirt road

The strip of highway has length $9 - x$. At a speed of 55 mph, it will take

$\dfrac{9-x}{55}$ hours to traverse the strip of highway

3.4. Optimization Problems



Graph of time of trip as function of x.

Step 2. Find the function and the interval

The total number of hours for the trip is

$$\boxed{T(x)=\frac{\sqrt{16+x^2}}{20}+\frac{9-x}{55}}$$

Over which interval does the optimization take place?

Since the dirt road joins the highway somewhere between P and the city, we have $0\le x\le 9$.

Step 3. Optimize the function

Our problem is to find the minimum of $T(x)$ on **[0, 9]**.

We solve $T'(x)=0$ to find the critical points: $T'(x)=\dfrac{x}{20\sqrt{16+x^2}}-\dfrac{1}{55}=0$

$$55x=20\sqrt{16+x^2}$$
$$11x=4\sqrt{16+x^2}$$
$$121x^2=16(16+x^2)$$

Thus, $x=16/\sqrt{105}\approx 1.56$ miles.

To find the minimum value of $T(x)$, we compute $T(x)$

at the critical point and endpoints of [0, 9]:

$T(0)\approx 0.36$ h

$T(1.56)\approx 0.35$ h

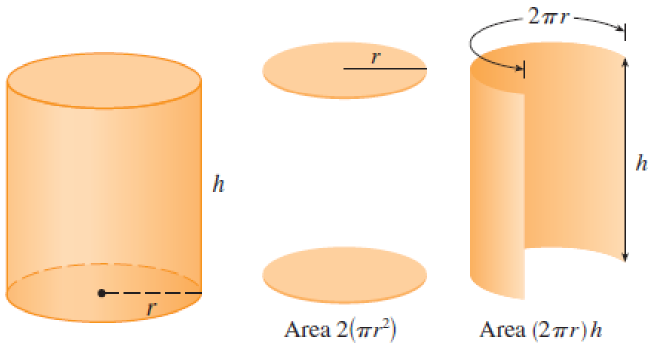
$T(9)\approx 0.49$ h

We conclude that the travel time is minimized if the dirt road joins the highway at a distance $x\approx 1.56$ miles from P.

3.4. Optimization Problems

2. Economical example (minimizing cost)

A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



Step 1. Choose variables

r is the radius and h the height (both in centimeters).

In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides).

The sides are made from a rectangular sheet with dimensions $2\pi r$ and h.

So the surface area is $A = 2\pi r^2 + 2\pi r h$

To eliminate h we use the fact that the volume is given as 1 L, which we take to be 1000 cm^3.

$\pi r^2 h = 1000$ which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression A for gives

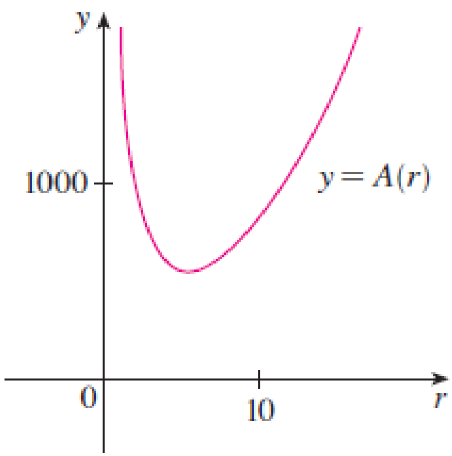
$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

3.4. Optimization Problems

Step 2. Find the function and the interval

Therefore the function that we want to minimize is

$$A = 2\pi r^2 + \frac{2000}{r}, \quad r > 0$$



Since the domain of A is $(0;\infty)$.

We can't use the argument concerning endpoints.

Step 3. Optimize the function

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

We can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$

So A is decreasing for all r to the left of the critical number and increasing for all r to the right.

$r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum. $h = \dfrac{1000}{\pi r^2} = \dfrac{1000}{\pi(500/\pi)^{2/3}} = 2\cdot\sqrt[3]{\dfrac{500}{\pi}} = 2r$

To minimize the cost of the can, the radius should be approximately 5.42 cm and the height should be equal to twice the radius, namely, the diameter.

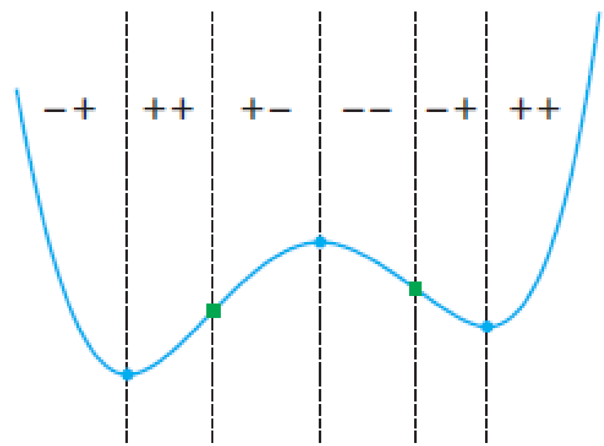
3.5. Graphing Functions

When sketching the graph $y = f(x)$ of a function f, we have three sources of useful information:

1. **the function f itself**, from which we determine the coordinates of some points on the graph, the symmetry of the graph, and any asymptotes;
2. **the first derivative, f'**, from which we determine the intervals of increase and decrease and the location of any local extreme values; and
3. **the second derivative, f''**, from which we determine the concavity and inflection points, and sometimes extreme values.

f'' \ f'	+ Concave up	− Concave down
+ Increasing	+ +	+ −
− Decreasing	− +	− −

Most graphs are made up of smaller arcs that have one of the four basic shapes, corresponding to the four possible sign combinations of f' and f''.

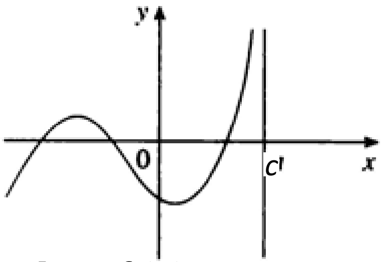


We pay particular attention to the **transition points**, where the basic shape changes due to a sign change in either f' (local min or max) or f'' (point of inflection).

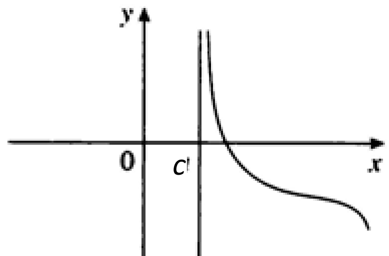
3.5. Graphing Functions

Def. Vertical Asymptote (recall!)

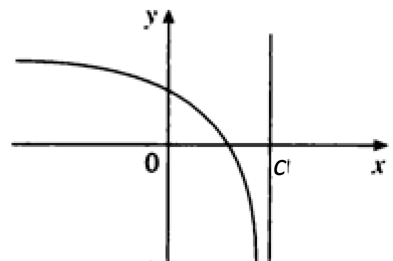
The line $x = c$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:



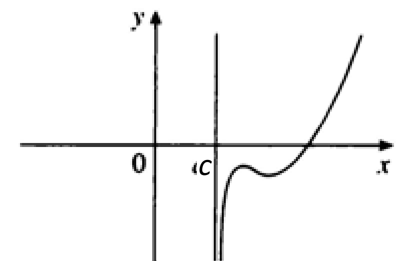
$\lim_{x \to c^-} f(x) = \infty$



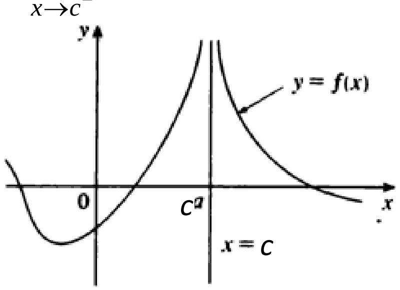
$\lim_{x \to c^+} f(x) = \infty$



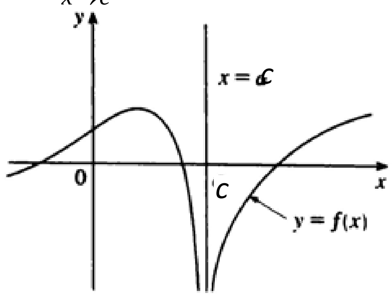
$\lim_{x \to c^-} f(x) = -\infty$



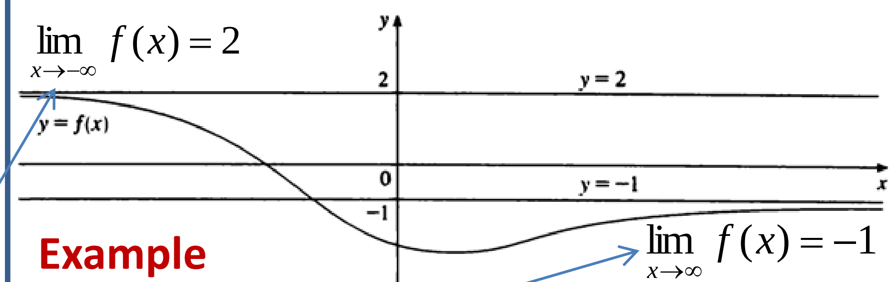
$\lim_{x \to c^+} f(x) = -\infty$



$\lim_{x \to c} f(x) = \infty$



$\lim_{x \to c} f(x) = -\infty$



$\lim_{x \to -\infty} f(x) = 2$

$\lim_{x \to \infty} f(x) = -1$

Example

Def. Horizontal Asymptote (recall!)

The line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either

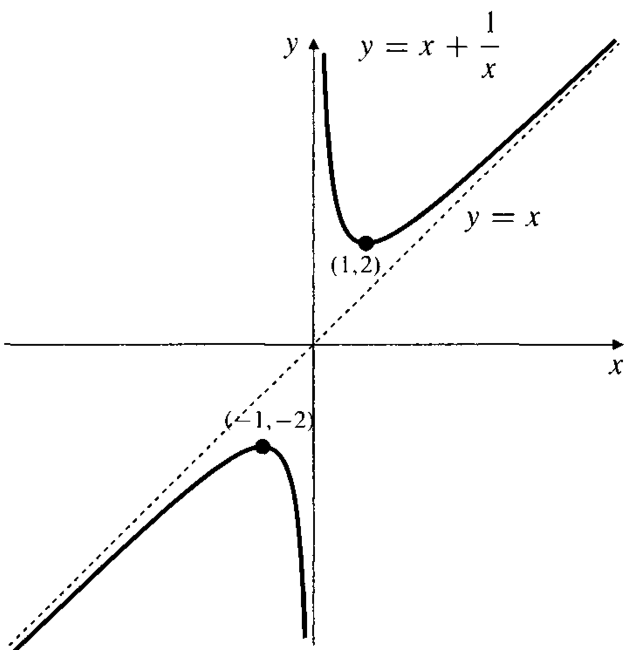
$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

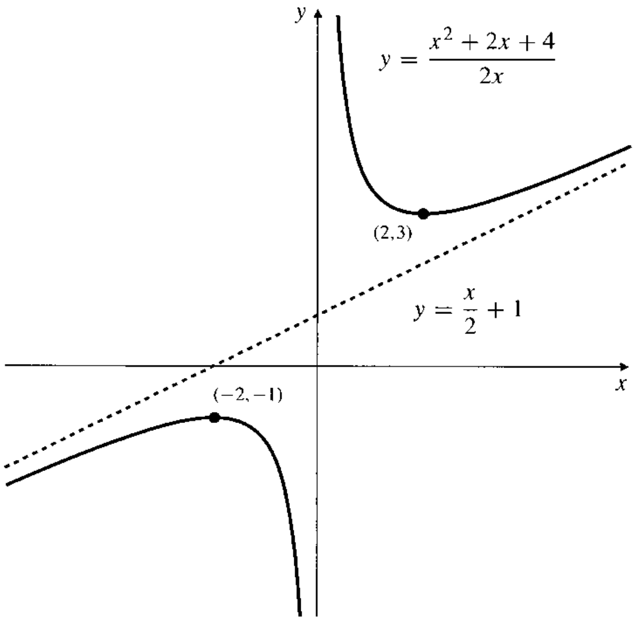
3.5. Graphing Functions

Def. Oblique Asymptote

The straight line $y=ax+b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if either

$$\lim_{x\to-\infty}(f(x)-(ax+b))=0 \quad \text{or} \quad \lim_{x\to\infty}(f(x)-(ax+b))=0, \quad \text{or both.}$$





$$a = \lim_{x\to\pm\infty}\frac{f(x)}{x}$$

$$b = \lim_{x\to\pm\infty}[f(x)-ax]$$

$$y = ax+b$$

It can happen that the graph of a function ***f*** approaches a nonhorizontal straight line as x approaches ∞ or $-\infty$ (or both).

3.5. Graphing Functions

Polynomial function. Sketch the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3.$

Step 1. Precalculus analysis

a) *The domain :* x - any real numbers.

b) *Symmetry:* this function is neither even nor odd, so we do not have any of the usual symmetries.

c) *The intercepts:* the y-intercept $f(0) = 3$, when $x = 0$.

Step 2. Calculus analysis

a) *The first derivative:* set the derivative equal to zero to find the critical points:

$f'(x) = x^2 - x - 2 = (x+1)(x-2) = 0.$ The critical points $c = -1, 2$ divide the x-axis into intervals

$(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$:

Interval	Test Value	Sign of f'
$(-\infty, -1)$	$f'(-2) = 4$	$+$
$(-1, 2)$	$f'(0) = -2$	$-$
$(2, \infty)$	$f'(3) = 4$	$+$

b) *The second derivative:* set the second derivative equal to zero and solve:

$f''(x) = 2x - 1 = 0.$ It has the solution $c = ½$ and the sign of the second derivative is as follows:

Interval	Test Value	Sign of f''
$(-\infty, \frac{1}{2})$	$f''(0) = -1$	$-$
$(\frac{1}{2}, \infty)$	$f''(1) = 1$	$+$

3.5. Graphing Functions

Polynomial function. Sketch the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3.$

Step 2. Calculus analysis

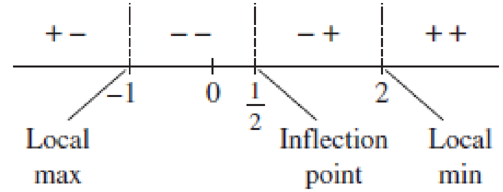
c) Note transition points and sign combinations:

There are three transition points:

$c = -1$: local max since $f\,'$ changes from $+$ to $-$ at $c = -1$.

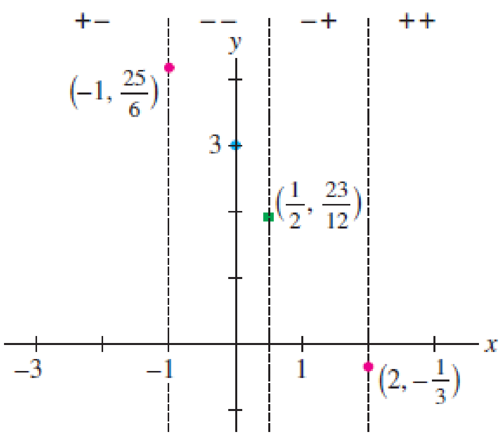
$c = ½$: point of inflection since $f\,''$ changes sign at $c = ½$.

$c = 2$: local min since $f\,'$ changes from $-$ to $+$ at $c = 2$.

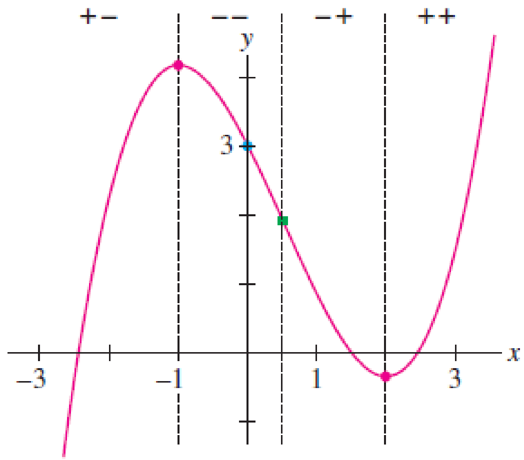


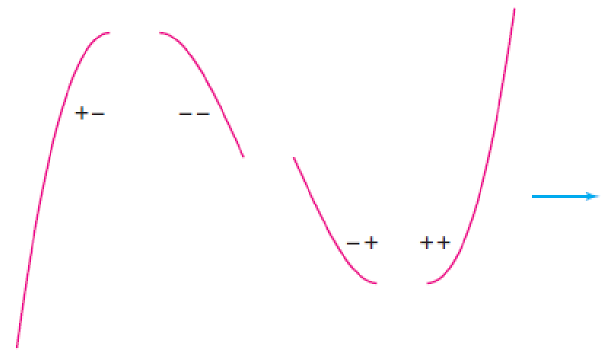
It is necessary to compute y-values at these points:
$f(-1) = 25/6$; $f(1/2) = 23/12$; $f(2) = -1/3$.

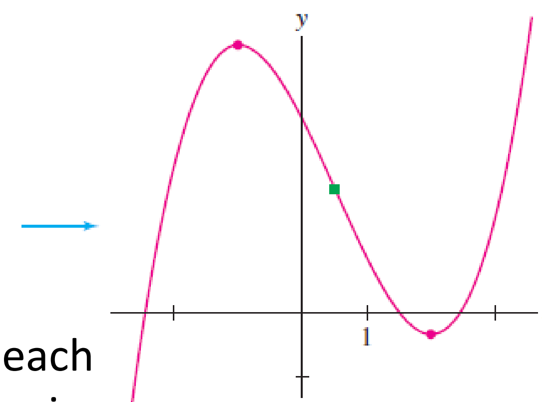
Step 3. Plot a few points



Step 4. Sketch the graph







Notice that the graph of our cubic is built out of four arcs, each with the appropriate increase/decrease and concavity behavior

3.5. Graphing Functions

Trigonometric function. Sketch the graph of $f(x) = \cos x + \frac{1}{2}x$ over the interval $[0,\pi]$.

Step 1. Precalculus analysis

a) The domain : x in $[0,\pi]$.

b) Symmetry: this function is neither even nor odd, so we do not have any of the usual symmetries.

c) The intercepts: the y-intercept $f(0) = 1$, when $x = 0$.

Step 2. Calculus analysis

a) The first derivative: $f'(x) = -\sin x + \frac{1}{2} = 0 \Rightarrow \quad x = \frac{\pi}{6}, \ \frac{5\pi}{6}$

Interval	Test Value	Sign of f'
$(0, \frac{\pi}{6})$	$f'(\frac{\pi}{12}) \approx 0.24$	+
$(\frac{\pi}{6}, \frac{5\pi}{6})$	$f'(\frac{\pi}{2}) = -\frac{1}{2}$	−
$(\frac{5\pi}{6}, \pi)$	$f'(\frac{11\pi}{12}) \approx 0.24$	+

End points

b) The second derivative: $f''(x) = -\cos x = 0 \Rightarrow \quad x = \frac{\pi}{2}$

Interval	Test Value	Sign of f''
$(0, \frac{\pi}{2})$	$f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$	−
$(\frac{\pi}{2}, \pi)$	$f''(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}$	+

End points

c) Note transition points and sign combinations:

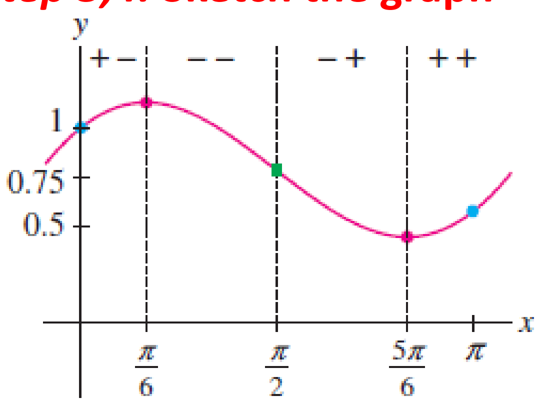
There are three transition points:

$c = \pi/6$: local max since f' changes from + to − at $c = \pi/6$.

$c = \pi/2$: point of inflection since f'' changes sign at $c = \pi/2$.

$c = 5\pi/6$: local min since f' changes from − to + at $c = 5\pi/6$.

Step 3,4. Sketch the graph



It is necessary to compute y-values at these points:
$f(\pi/6) \approx 1.13$; $f(\pi/2) \approx 0.79$; $f(5\pi/6) \approx 0.44$; $f(\pi) \approx 0.57$.

3.5. Graphing Functions

Rational function. Sketch the graph of $f(x)=\dfrac{x^2+2x+4}{2x}$.

$$f(x)=\frac{x^2+2x+4}{2x}=\frac{x}{2}+1+\frac{2}{x}$$

Step 1. Precalculus analysis

a) The domain : all x except 0.

b) Symmetry: none obvious (y is neither odd nor even).

c) The intercepts: none. $x^2+2x+4=(x+1)^2+3\ge 3$ for all x, and y is not defined at $x=0$.

Step 2. Calculus analysis

a) The first derivative: $f'(x)=\dfrac{1}{2}-\dfrac{2}{x^2}=\dfrac{x^2-4}{2x^2}$

b) The second derivative: $f''(x)=\dfrac{4}{x^3}$

$f''=0$ nowhere; f'' undefined at $x=0$.

		Stationary point		*Singular point*		*Stationary point*	
x		-2		0		2	
y'	$+$	0	$-$	undef	$-$	0	$+$
y''	$-$		$-$	undef	$+$		$+$
y	$\nearrow$	max	$\searrow$	undef	$\searrow$	min	$\nearrow$
	⌢		⌢		⌣		⌣

no point of inflection, f'' changes sign at 0, but f'' DNE at *0*

c) Note transition points and sign combinations:

There are three transition points:

$c=-2$: local max since f' changes from $+$ to $-$ at $c=-2$.

$c=2$: local min since f' changes from $-$ to $+$ at $c=2$.

It is necessary to compute y-values at these points: $f(-2)=-1$; $f(2)=3$.

d) Asymptotes:

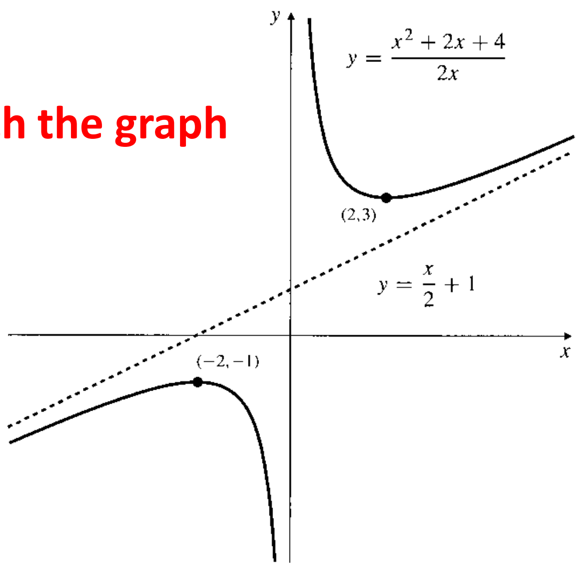
Vertical asymptote: $x=0$, $\lim\limits_{x\to 0}\dfrac{x^2+2x+4}{2x}=\infty$

Horizontal asymptote: DNE

Oblique asymptote: $y=\dfrac{x}{2}+1$

$$\lim_{x\to\pm\infty}\Bigg[\underbrace{\frac{x}{2}+1+\frac{2}{x}}_{f(x)}-\underbrace{\left(\frac{x}{2}+1\right)}_{ax+b}\Bigg]=\lim_{x\to\pm\infty}\frac{2}{x}=0$$

Step 3,4. Sketch the graph



3.5. Graphing Functions

Step 1. Precalculus analysis

a) *Domain*
b) *Symmetry*
c) *Intercepts*

Step 2. Calculus analysis

a) *The first derivative*
b) *The second derivative*
c) *Transition points and sign combinations*
d) *Asymptotes*

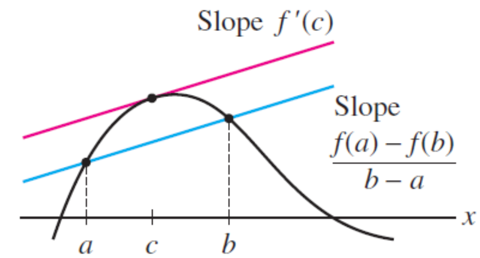
Step 3. Plot a few points

Step 4. Sketch the graph

3.6. The Mean Value Theorem for Derivatives

Consider the secant line through points $(a, f(a))$ and $(b, f(b))$ on a graph.

You can see there exists at least one tangent line that is parallel to the secant line in the interval (a, b).



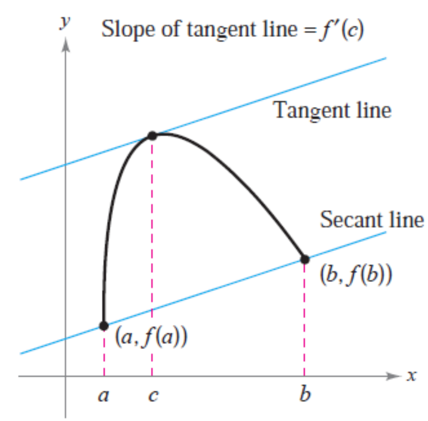
Because two lines are parallel if they have the same slope, what the Mean Value Theorem claims is that there exists a point c between a and b such that

$$\underbrace{f'(c)}_{\text{Slope of tangent line}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{Slope of secant line}}$$

Theorem A. Mean Value Theorem for Derivatives

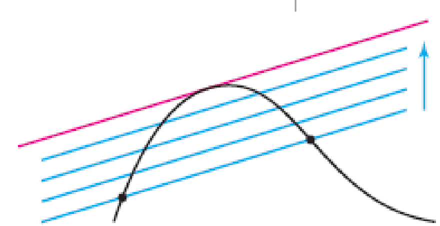
If f is **continuous** on a closed interval $[a, b]$ and **differentiable** on its interior (a, b), then there is at least one number c in (a, b) where

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$



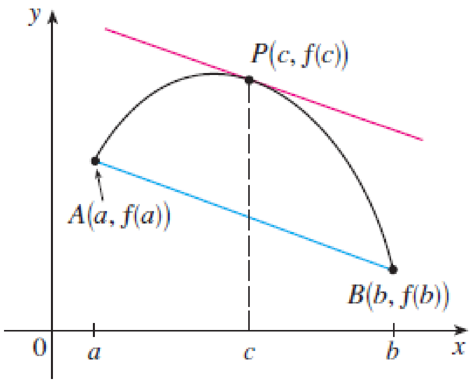
or equivalently, where $f(b) - f(a) = f'(c)(b-a)$

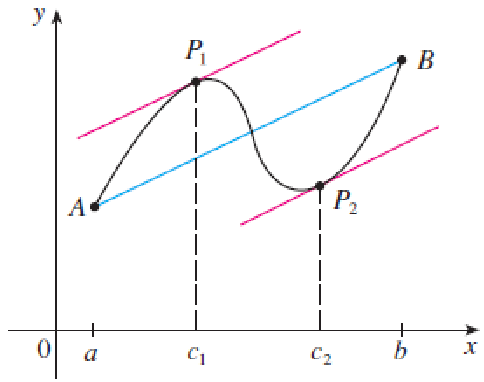
To see that the Mean Value Theorem is plausible, imagine what happens when the secant line is moved parallel to itself.

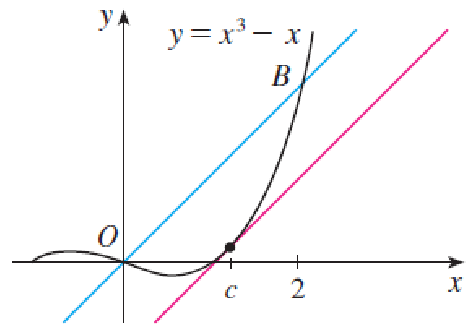


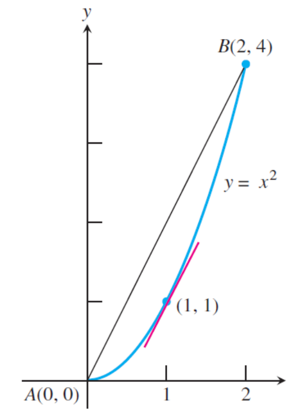
3.6. The Mean Value Theorem for Derivatives

Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the *tangent line is parallel to the secant line AB.*



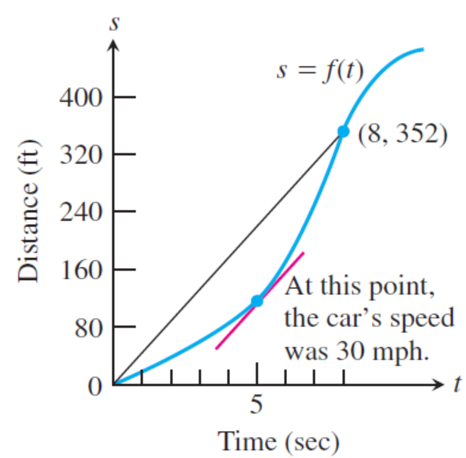






Physical Interpretation: If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the *instantaneous change* at some interior point must equal the *average change* over the entire interval.

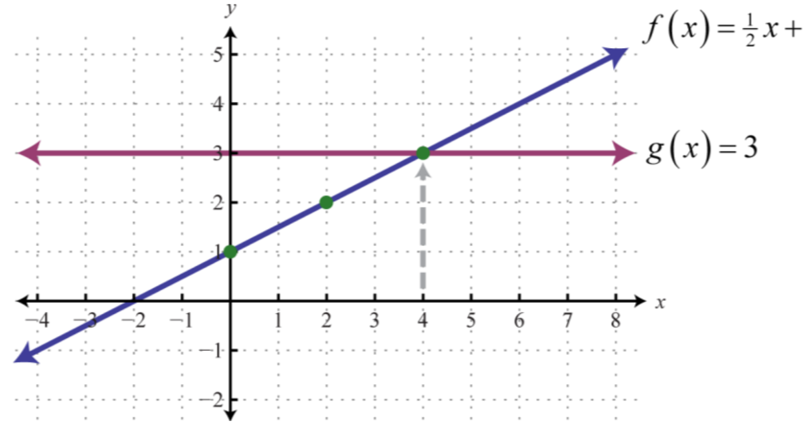
- If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is 352/8= 44 ft/sec, or 30 mph.
- At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph.



3.6. The Mean Value Theorem for Derivatives

Corollary 1. If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Functions with f ' = 0 are Constant



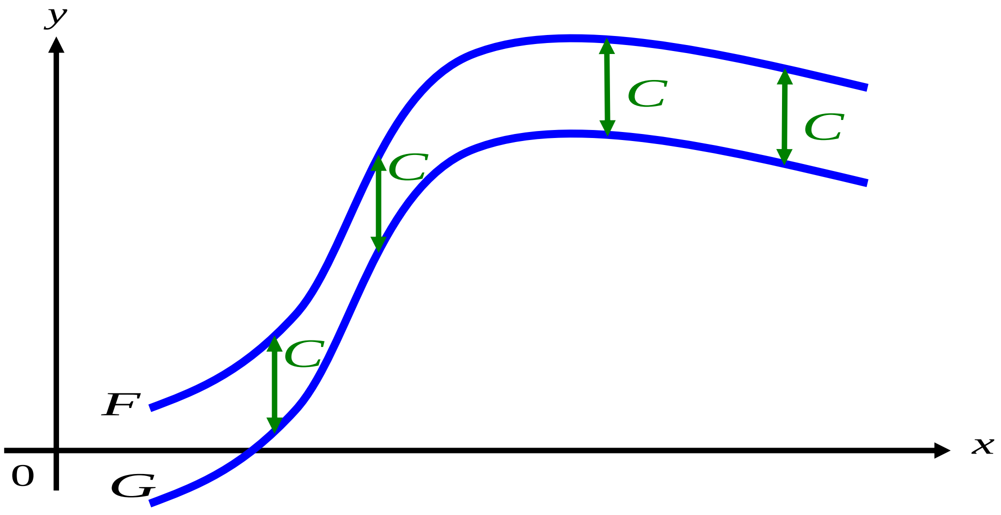
Theorem B. (Corollary 2.)

If $F'(x) = G'(x)$ for all x in an interval (a, b), then there is a constant C such that

$$\boxed{F(x) = G(x) + C}$$

for all x in (a, b).

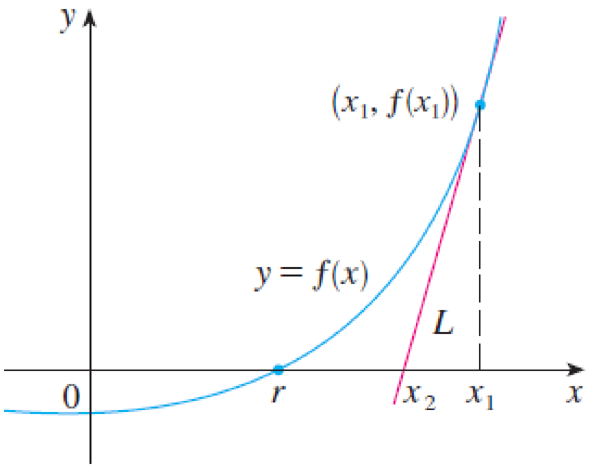
Functions with the Same Derivative Differ by a Constant



3.7. Solving Equations Numerically

Newton's Method

- Suppose the root that we are trying to find is labeled r.
- We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of f.
- Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.
- *The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking).*



- To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x_2 - x_1)$$

- Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

- If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

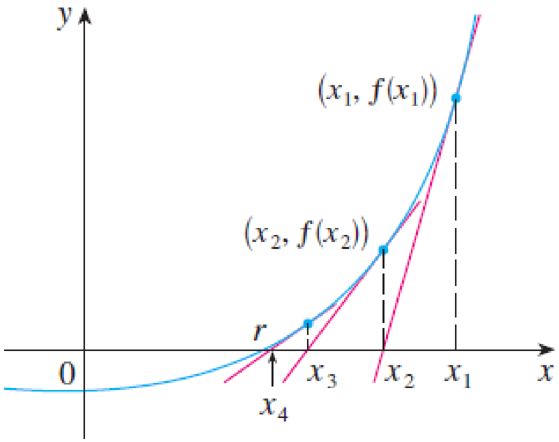
- We use x_2 as a second approximation to r.

The Mean Value Theorem

$$\boxed{\frac{f(b) - f(a)}{b - a} = f'(c)}$$

3.7. Solving Equations Numerically

Newton's Method



- Next we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation: $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
- If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$
- In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method To find a numerical approximation to a root of $f(x) = 0$:

Step 1. Choose initial guess x_0 (close to the desired root if possible).

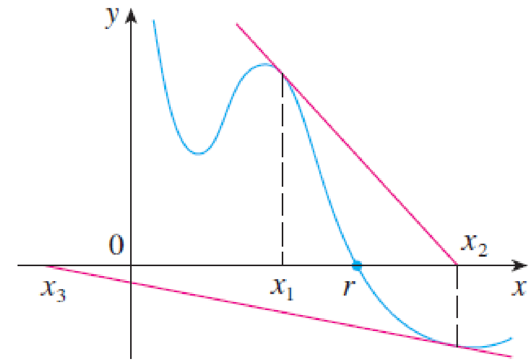
Step 2. Generate successive approximations $x_1, x_2, \ldots$ where

recursion formula or iteration scheme

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- If the numbers x_n become closer and closer to r as becomes large, then we say that the sequence *converges* to r and we write $\lim_{n\to\infty} x_n = r$

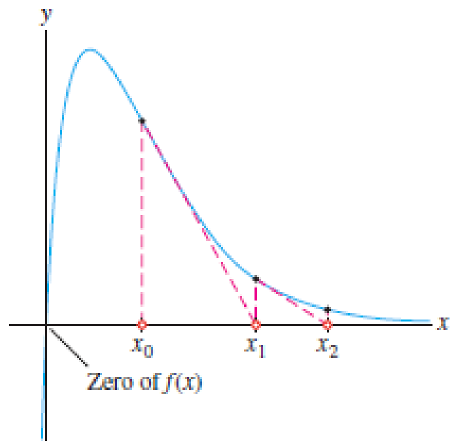
3.7. Solving Equations Numerically



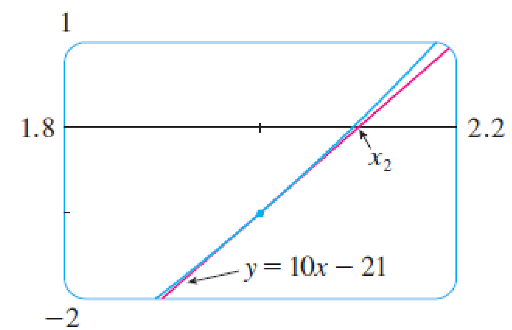
- Although the sequence of successive approximations converges to the desired root for functions, in certain circumstances the sequence may not converge.
- You can see (figure) that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in figure) falls outside the domain of f.
- ***Then Newton's method fails and a better initial approximation should be chosen.***

How Many Iterations Are Required?

In practice, it is usually safe to assume that if x_n and x_{n+1} agree to m decimal places, then the approximation x_n is correct to these m places.



Function has only one zero but the sequence of Newton iterates goes off to infinity.



the geometry behind the first step in Newton's method

8.1. Indeterminate Forms of Type 0/0

Theorem A. L'Hôpital's Rule for forms of type 0/0

Suppose that $\lim_{x\to u} f(x) = \lim_{x\to u} g(x) = 0.$

If $\lim_{x\to u}[f'(x)/g'(x)]$ exists in either the finite or infinite sense (i.e., if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\boxed{\lim_{x\to u}\frac{f(x)}{g(x)} = \lim_{x\to u}\frac{f'(x)}{g'(x)}}$$

- L'Hôpital's Rule says that *the limit of a quotient of functions is equal to the limit of the quotient of their derivatives*, provided that the **given conditions** are satisfied.
- **Notice that when using l' Hôpital's Rule we differentiate the numerator and denominator separately.**

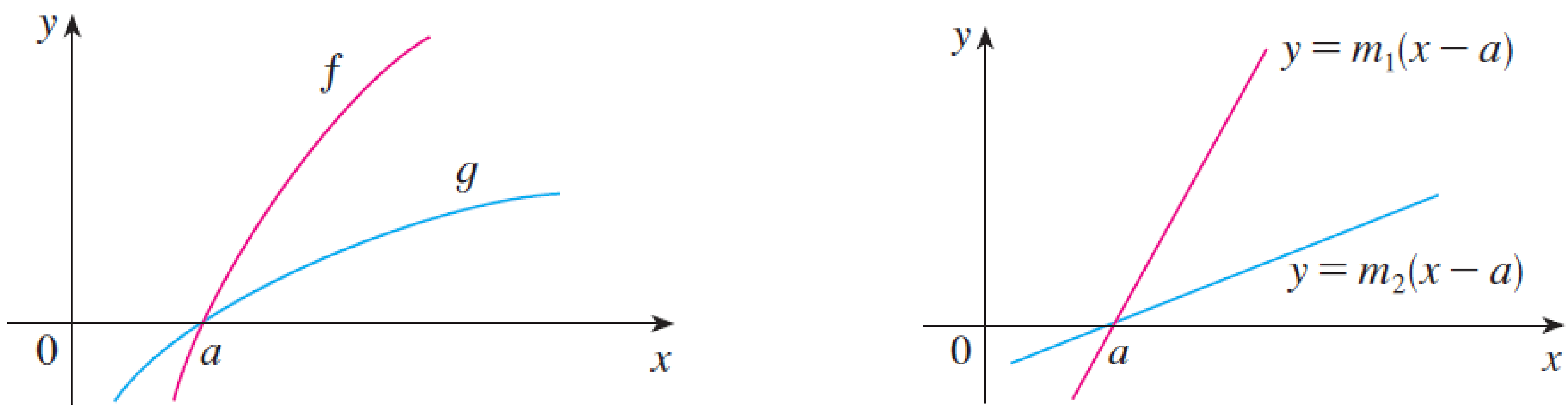
We do not use the Quotient Rule !!!

- L'Hôpital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

GUILLAUME L'HÔPITAL (1661–1704)

L'Hôpital's Rule is named after the French mathematician Guillaume François Antoine de L'Hôpital. L'Hôpital is credited with writing the first text on differential calculus (in 1696) in which the rule publicly appeared. It was recently discovered that the rule and its proof were written in a letter from John Bernoulli to L'Hôpital. "… I acknowledge that I owe very much to the bright minds of the Bernoulli brothers. … I have made free use of their discoveries …," said L'Hôpital.

8.1. Indeterminate Forms of Type 0/0



This figure suggests visually why l'Hôpital's Rule might be true.

- The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a,0)$, the graphs would start to look almost linear. But if the functions actually were linear, as in the second graph, then their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Approximations and numerical calculations

Linear approximation

Assume that f is differentiable at $x = a$. If x is close to a, then $L(x) = f'(a)(x-a) + f(a)$ is the *linearization* of f at $x = a$. The Linear Approximation can be rewritten as the estimate $f(x) \approx L(x)$ for small $|x - a|$.
$f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\Delta x$ - a good approximation for finding roots and powers of the numbers.

Solving equations numerically

Newton's Method: To find a numerical approximation to a root of $f(x) = 0$:

Iteration scheme :

Step 1. Choose initial guess x_0 (close to the desired root if possible).

Step 2. Generate successive approximations $x_1, x_2, \ldots$ where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Finding limits

L'Hôpital's Rule says that *the limit of a quotient of functions is equal to the limit of the quotient of their derivatives*, provided that some conditions are satisfied.

$$\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}$$

Mean Value Theorem

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

Corollary: Functions with the Same Derivative Differ by a Constant

Graphing functions

Monotonicity

Let f be *continuous* on an interval I and *differentiable* at every interior point of I. We say that

If $f'(x)>0$ for all x interior to I, then f is **increasing** on I.

If $f'(x)<0$ for all x interior to I, then f is **decreasing** on I.

Concavity

Let f be *twice differentiable* on the open interval I.

If $f''(x)>0$ for all x in I, then f is **concave up** on I.

If $f''(x)<0$ for all x in I, then f is **concave down** on I.

If $f''(x)=0$ and $f''(x)$ changes sign at c, then c is a **point of inflection.**

Extrema

(First derivative test) Let f be continuous on an open interval (a, b) that contains a critical point c.

If $f'(x)>0$ for all x in (a, c) and $f'(x)<0$ for all x in (c, b), then $f(c)$ is **a local maximum** of f.

If $f'(x)<0$ for all x in (a, c) and $f'(x)>0$ for all x in (c, b), then $f(c)$ is **a local minimum** of f.

(Second derivative test) Suppose that $f'(c)=0$.

If $f''(x)<0$, then $f(c)$ is **a local maximum** of f.

If $f''(x)>0$, then $f(c)$ is **a local minimum** of f.

$$\int f(x)dx = F(x) + C$$

Antiderivative

Practical problems

Related rate problems

The goal is to calculate an unknown rate of change in terms of other rates of change that are known. This will usually require implicit differentiation.

Algorithm:

Step 1. Assign variables and restate the problem.

Step 2. Find an equation that relates the variables and differentiate.

Step 3. Use the given data to find the unknown derivative.

Optimization problems

We are required to find the optimal (best) way of doing something. This will usually require extrema theorems.

Algorithm:

Step 1. Choose variables.

Step 2. Find the function and the interval.

Step 3. Optimize the function.

Differential equations

Any equation in which the unknown is a function and that involves derivatives of this unknown function.

Algorithm:

Step 1. Choose variables.

Step 2. Write the differential equation and separate variables.

Step 3. Evaluate the solution.

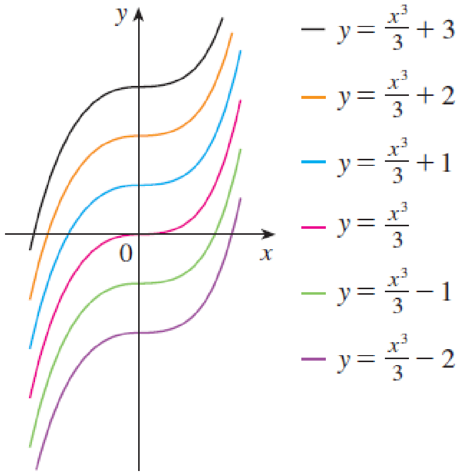
$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\overline{x_i})\Delta x_i$$

Definite Integral

3.8. Antiderivatives: introduction

Def. Antiderivatives

We call F **an** *antiderivative* of f on the interval I if $D_x F(x) = f(x)$ on I that is, if $F'(x) = f(x)$ for all x in I.



Members of the family of antiderivatives of $f(x) = x^2$

We used the *Mean Value Theorem* to prove that if *two functions have identical derivatives on an interval, then they must differ by a constant.*

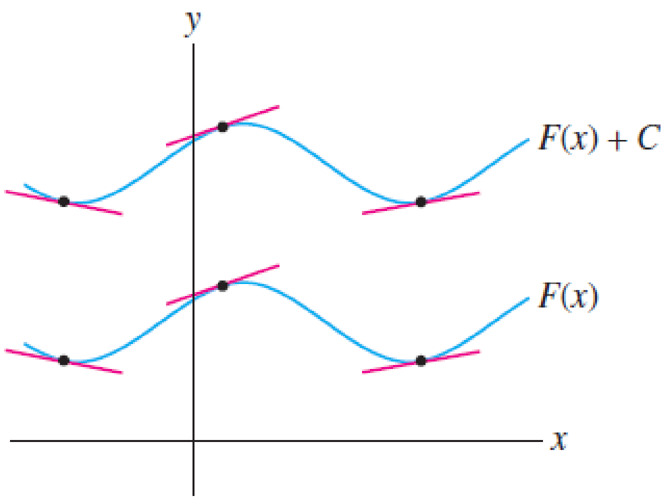
Thus if F and G are any two antiderivatives of f, then $F'(x) = f(x) = G'(x)$

So, $G(x) - F(x) = C$ where is a constant. We can write this as

$G(x) = F(x) + C$

Def. The General Antiderivative

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.



The tangent lines to the graphs of $y = F(x)$ and $y = F(x) + C$ are parallel. Vertical shifting moves the tangent lines without changing their slopes.

3.8. Antiderivatives: introduction

Notation for Antiderivatives

The notation $y = \int f(x)dx = F(x) + C$ means that F is an antiderivative of f on an interval.

The operation of finding all antiderivatives for the function is called **antidifferentiation (or integration)**

the integral sign

variable of integration

an antiderivative of *f (x)*

integrand

$$\boxed{\int f(x)dx = F(x) + C}$$

constant of integration

The expression $\int f(x)dx$ is read as the *antiderivative of f with respect to x.*

So, the differential dx serves to identify x as the variable of integration.

The term **indefinite integral (**as well as **primitive function)** is a synonym for antiderivative.

3.8. Antiderivatives: Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\boxed{\int F'(x)dx = F(x) + C}$$

Integration is the "inverse" of differentiation

Moreover, if $y = \int f(x)dx = F(x) + C$ then differentiating both sides yields

$$\boxed{D_x \int f(x)dx = f(x)}$$

Differentiation is the "inverse" of integration

$$\boxed{\int dx = x + C}$$

3.8. Antiderivatives: Basic Integration Rules

Differentiation Formulas	*Integration Formulas* ***(Theorems A,B,C)***
$(kf)' = k \cdot f', \;\; k = const$	$\int kf(x)dx = k\int f(x)dx, \;\; k = const$
	the antiderivative of a constant times a function is the constant times the antiderivative of the function (a constant multiplier can be passed across indefinite integral)
$(f+g)' = f' + g'$	$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
$(f-g)' = f' - g'$	$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$
	the antiderivative of a sum (difference) is the sum (difference) of the antiderivatives
$(\sin x)' = \cos x$	$\int \sin x dx = -\cos x + C$
$(\cos x)' = -\sin x$	$\int \cos x dx = \sin x + C$
$(x^n)' = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \;\; n \neq 1$ Power rule
	to integrate a power of *x*, we increase the exponent by 1 and divide by the new exponent

3.8. Antiderivatives: Basic Integration Rules

Recall *the chain rule* as applied to a power of a function.

$u=g(x)$ is a differentiable function and r is a rational number ($r \neq -1$)

$$D_x\left[\frac{u^{r+1}}{r+1}\right]=u^r \cdot D_x u \;\Rightarrow\; D_x\left[\frac{[g(x)]^{r+1}}{r+1}\right]=[g(x)]^r \cdot g'(x)$$

Theorem D. Generalized Power Rule

Let g be a differentiable function and r is a rational number different from -1.

Then

$$\boxed{\int [g(x)]^r g'(x)dx = \frac{[g(x)]^{r+1}}{r+1}+C}$$

If we let $u=g(x)$ then $du = g'(x)dx$

$$\int u^r du = \frac{u^{r+1}}{r+1}+C, \;\; r \neq 1$$

3.9. Intro to Differential Equations

Let's antdifferentiate (integrate) a *f* function to obtain a new function *F*

$$\int f(x)dx = F(x) + C$$

$F'(x) = f(x)$ is equivalent (in differential notation) to $dF(x) = f(x)dx$

$$\boxed{\int dF(x) = F(x) + C}$$

We integrate the differential of a function to obtain the function (plus a constant)

A **differential equation** is any equation in which the unknown is a function and that involves derivatives of this unknown function

- A function that, when substituted in the differential equation yields an equality, is called a **solution** of the differential equation
- To solve a differential equation is **to find** an **unknown function**
- **First-order separable differential equations** are equations involving just the first derivative of the unknown function and are such that the variables can be separated, one on each side of the equation

3.9. Intro to Differential Equations

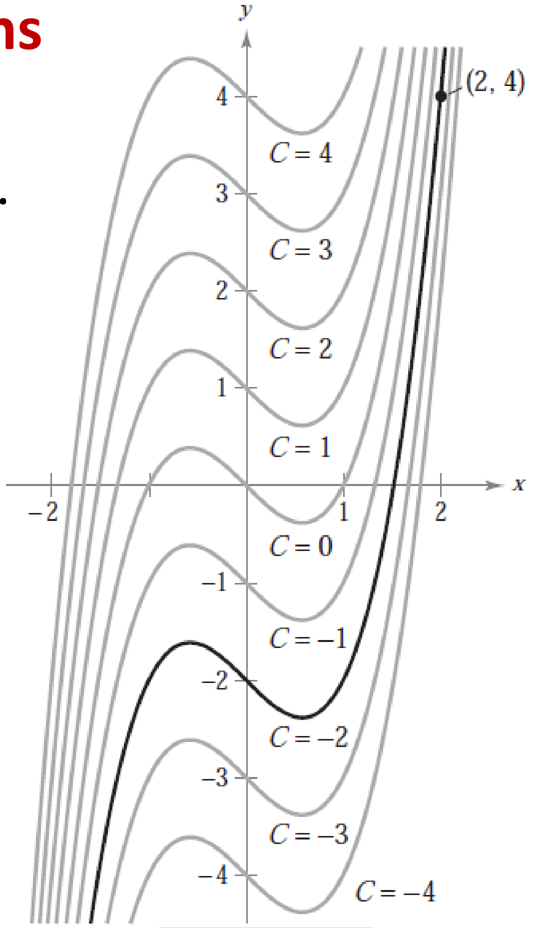
In many applications of integration, you are given enough information to determine a **particular solution.** This information is called an **initial condition.**

$$\frac{dy}{dx} = 3x^2 - 1 \qquad F(x) = x^3 - x + C$$ General solution

$$F(2) = 4$$ Initial condition

$$F(2) = 8 - 2 + C = 4 \ \Rightarrow C = -2$$

$$F(x) = x^3 - x - 2$$ Particular condition



$F(x) = x^3 - x + C$

The particular solution that satisfies the initial condition $F(2) = 4$ is

$$F(x) = x^3 - x - 2$$

Motion Problems

$$v(t) = s'(t) = \frac{ds}{dt} \qquad s(t) = \int v(t)dt$$ *Position is an antiderivative of velocity*

$$a(t) = v'(t) = \frac{dv}{dt} \qquad v(t) = \int a(t)dt$$ *Velocity is an antiderivative of acceleration*