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ON LINEARLY INDEPENDENT SOLUTIONS OF THE HOMOGENEOUS SCHWARZ PROBLEM

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Abstract. We study the homogeneous Schwarz problem for Douglis analytic functions. We consider two-dimensional matrices J with a multiple eigenvalue and a eigenvector, which is not proportional to a real vector. We obtain a sufficient condition for the matrix J under which there exist two linearly independent solutions of the problem defined in a certain domain D. We present an example.

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1. Basic definitions and statement of the problem. Assume that a matrix $J \in \mathbb{C}^{n\times n}$ has no real eigenvalues. Let $\omega = \omega(z) \in C^1(D)$ be an n-vector-valued function, where $D \subset \mathbb{R}^2$ is a domain. Let us consider the following homogeneous elliptic system of first-order partial differential equations in D (see [3, 6, 7]):

$$\frac{\partial \omega}{\partial y} - J\frac{\partial \omega}{\partial x} = 0, \quad z \in D. \tag{1}$$

Definition 1 (see [2, 3, 5–7])**.** A function $\omega(z)$ considered as a solution of (1) is called a *Douglis analytical* function or a J-analytical function. We say that the function $\omega(z)$ corresponds to the matrix J.

A proof of the fact that the system (1) is elliptic can be found in [5]. Examples of J-analytical functions are vector polynomials of the form

$$\omega(z) = \sum_{k=0}^{m} (xE + yJ)^k \cdot c_k, \quad c_k \in \mathbb{C}^n,$$

where E is the identity matrix.

Let us consider the following homogeneous Schwarz problem for the system (1) (see [2, 3, 6, 7]).

Let a simply connected domain $D \subset \mathbb{R}^2$ be bounded by a smooth contour Γ. Find a J-analytical function $\omega(z) \in C(\overline{D})$ with the matrix J satisfying the the boundary condition

$$\operatorname{Re} \omega(z)\big|_{\Gamma} = 0. \tag{2}$$

The obvious solutions of the problem (2) are constant vectors $\omega \equiv ic$, where $c \in \mathbb{R}^n$, which are called *trivial* (constant) solutions. As is known (see [4]), only constants are solutions of the problem (2) for $n = 1$. However, this is invalid in general for $n > 1$. We present an example for $n = 2$. Let

$$J = \begin{pmatrix} 4i & 9 \\ 1 & -2i \end{pmatrix}, \quad \omega(z) = \begin{pmatrix} 2x^2 + 4y^2 - 1 - 2i\,xy \\ -i(x^2 + y^2) \end{pmatrix}. \tag{3}$$

In (3), the matrix J has the eigenvalue $\lambda = i$ of multiplicity 2. The function $\omega(z)$ corresponds to the matrix J according to Definition 1. Here $\operatorname{Re} \omega(z)\big|_{\Gamma} = 0$ on the ellipse $\Gamma : 2x^2 + 4y^2 = 1$.

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Before constructing nonconstant solutions of the problem (2) for $n = 2$ in domains *other than ellipses*, we reduce it to an equivalent scalar form.

2. Scalar form of the homogeneous Schwarz problem for $n = 2$ and $\lambda = i$. Assume that a matrix $J \in \mathbb{C}^{2\times 2}$ has a multiple eigenvalue $\lambda = i$. We denote by J_1 and Q, respectively, the Jordan form and Jordan basis of the matrix J:

$$J_1 = \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix}, \quad Q = (\boldsymbol{x}, \boldsymbol{y}), \quad J = QJ_1Q^{-1}. \tag{4}$$

Moreover, we assume that the eigenvector $\boldsymbol{y}$ *is not proportional to a real vector.* We decompose the *complex conjugate* vector $\overline{\boldsymbol{y}}$ with respect to the Jordan basis $\boldsymbol{x}, \boldsymbol{y}$ of the matrix J:

$$\overline{\boldsymbol{y}} = l_1\boldsymbol{x} + l_2\boldsymbol{y}, \quad l_1, l_2 \in \mathbb{C}, \quad l_1 = l_1(J) = \frac{\det(\overline{\boldsymbol{y}}, \boldsymbol{y})}{\det(\boldsymbol{x}, \boldsymbol{y})}, \quad l_2 = l_2(J) = \frac{\det(\boldsymbol{x}, \overline{\boldsymbol{y}})}{\det(\boldsymbol{x}, \boldsymbol{y})}. \tag{5}$$

In (5), we find the numbers l_1 and l_2 by the Cramer formulas. Below we need only the number l_1. Let us prove the following property.

Proposition 1. *The absolute value $|l_1|$ of the number l_1 in* (5) *is independent of the choice of the Jordan basis Q of the matrix J.*

Proof. Since the matrix J has a unique eigenvector $\boldsymbol{y}$, any other its Jordan basis Q_1 can be written as follows:

$$Q_1 = (c\boldsymbol{x} + b\boldsymbol{y},\ a\boldsymbol{y}), \quad a, b, c \in \mathbb{C}, \quad a, c \neq 0.$$

It is easy to show that $c = a$, i.e.,

$$Q_1 = (\boldsymbol{x}_1, \boldsymbol{y}_1) = (a\boldsymbol{x} + b\boldsymbol{y},\ a\boldsymbol{y}), \quad a, b \in \mathbb{C}, \quad a \neq 0.$$

Assume that the number l_1^* was found by the formula (5), where instead of the basis Q we take the basis Q_1. Then, taking into account the properties of the determinant, we have

$$l_1^* = \frac{\det(\overline{\boldsymbol{y}}_1, \boldsymbol{y}_1)}{\det(\boldsymbol{x}_1, \boldsymbol{y}_1)} = \frac{\det(\overline{a\boldsymbol{y}}, a\boldsymbol{y})}{\det(a\boldsymbol{x} + b\boldsymbol{y}, a\boldsymbol{y})} = \frac{\det(\overline{a}\,\overline{\boldsymbol{y}},\ a\boldsymbol{y})}{\det(a\boldsymbol{x},\ a\boldsymbol{y})} = \frac{\overline{a}\,a}{a^2} \cdot \frac{\det(\overline{\boldsymbol{y}}, \boldsymbol{y})}{\det(\boldsymbol{x}, \boldsymbol{y})} = \frac{|a|^2}{a^2} \cdot l_1, \quad a \neq 0. \tag{6}$$

Therefore, $|l_1^*| = |l_1|$. □

Remark 1. It follows from (6) that the numbers l_1 and l_1^* themselves depend on the choice of the Jordan basis. They will be different if $a \notin \mathbb{R}$.

Let us consider a new basis $Q' = (\overline{\boldsymbol{y}}, \boldsymbol{y})$ of the operator J. Since $J\boldsymbol{x} = i\boldsymbol{x} + \boldsymbol{y}$, $J\boldsymbol{y} = i\boldsymbol{y}$, by virtue of (4), taking into account (5), we obtain

$$J\overline{\boldsymbol{y}} = J(l_1\boldsymbol{x} + l_2\boldsymbol{y}) = il_1\boldsymbol{x} + l_1\boldsymbol{y} + il_2\boldsymbol{y} = i(l_1\boldsymbol{x} + l_2\boldsymbol{y}) + l_1\boldsymbol{y} = i\overline{\boldsymbol{y}} + l_1\boldsymbol{y}. \tag{7}$$

Thus, the matrix $J_1' = (Q')^{-1}JQ'$ of the operator J in the new basis $Q' = (\overline{\boldsymbol{y}}, \boldsymbol{y})$ has the following form:

$$J_1' = \begin{pmatrix} i & 0 \\ l_1 & i \end{pmatrix}. \tag{8}$$

Substituting $J = Q'J_1'(Q')^{-1}$ in (1) and multiplying both sides of the resulting equation by the matrix $(Q')^{-1}$ from the left, we obtain the following relation:

$$\frac{\partial}{\partial y}\begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} i & 0 \\ l_1 & i \end{pmatrix} \cdot \frac{\partial}{\partial x}\begin{pmatrix} f \\ g \end{pmatrix} = 0, \quad (f, g)^T = (Q')^{-1}\omega, \quad Q' = (\overline{\boldsymbol{y}}, \boldsymbol{y}). \tag{9}$$

We introduce the complex variable $\zeta = x + iy$, $\overline{\zeta} = x - iy$. We take into account the following fact: if a function $f(\zeta)$ is holomorphic in a domain D, then $\dfrac{\partial f}{\partial x} = \dfrac{df}{d\zeta}$, $\zeta \in D$. Assume that $f(\zeta) = u + iv$. The following equalities follow from (9):

$$g(x,y) = l_1 \cdot y\,\frac{\partial f}{\partial x} + F_1 = l_1 \cdot \left(\frac{\zeta - \overline{\zeta}}{2i}\right)\frac{\partial f}{\partial x} + F_1 = \frac{il_1}{2} \cdot \overline{\zeta}\,\frac{\partial f}{\partial x} + F = l \cdot \overline{\zeta}\,\frac{df}{d\zeta} + F = p + iq, \tag{10}$$

where

$$F = \frac{l_1}{2i}\,\zeta\,\frac{df}{d\zeta} + F_1, \quad l = \frac{il_1}{2},$$

$f = f(\zeta)$, $F_1 = F_1(\zeta)$, $F = F(\zeta)$ are arbitrary holomorphic functions in D. In (10), u, v, p, and q denote the real-valued functions of the variables x and y.

Let a function $\omega = \omega(\zeta)$ be a solution of the homogeneous Schwarz problem (2) defined in a certain domain D. Let $\boldsymbol{y} = (a_1, a_2) = (a + bi,\, c + di)$, where $a, b, c, d \in \mathbb{R}$. Then taking into account (9), we represent the function $\omega(\zeta)$ in the form

$$\omega(\zeta) = Q' \cdot (f,g)^T = (\overline{\boldsymbol{y}}, \boldsymbol{y}) \cdot (f,g)^T = \begin{pmatrix} \overline{a}_1 & a_1 \\ \overline{a}_2 & a_2 \end{pmatrix} \cdot \begin{pmatrix} u + iv \\ p + iq \end{pmatrix}. \tag{11}$$

The boundary condition (2), i.e., the equality $\operatorname{Re}\omega(\zeta)\big|_\Gamma = 0$, takes the following form in the notation of (11):

$$\begin{cases} \operatorname{Re}\left[\overline{a}_1(u + iv) + a_1(p + iq)\right]\big|_\Gamma = 0, \\ \operatorname{Re}\left[\overline{a}_2(u + iv) + a_2(p + iq)\right]\big|_\Gamma = 0. \end{cases} \tag{12}$$

We consider (12) as an inhomogeneous algebraic system with respect to the variables u and v. Since, by the assumption, the eigenvector $\boldsymbol{y}$ of the matrix J *is not proportional to a real vector*, the determinant of this system is nonzero:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

In addition, the following identity holds for $k = 1, 2$:

$$\operatorname{Re}\left[\overline{a}_k(u + iv) + a_k(p + iq)\right]\Big|_{\substack{u=-p \\ v=q}} = \operatorname{Re}\left[\overline{a}_k(-p + iq) - \overline{\overline{a}_k(-p + iq)}\right] = 0. \tag{13}$$

Therefore, the unique solution of (12) with respect to the variables u and v has the following form:

$$u = -p, \quad v = q. \tag{14}$$

In turn, the pair of equalities (14) is equivalent to the complex equation

$$(p + iq) + (u - iv) = 0, \tag{15}$$

which *holds on the contour* Γ. Taking into account the notation (10) and (5), we rewrite Eq. (15) as follows:

$$l \cdot \overline{\zeta}\,\frac{df}{d\zeta} + \overline{f} + F\big|_\Gamma = 0, \quad f, F \in C^1(\overline{D}), \quad l = l(J) = \frac{i}{2}l_1 = \frac{i}{2}\,\frac{\det(\overline{\boldsymbol{y}}, \boldsymbol{y})}{\det(\boldsymbol{x}, \boldsymbol{y})}, \quad \overline{\zeta} = x - iy. \tag{16}$$

As a result, due to the invertibility of the transformations performed and taking into account (9), (10), and (8), we arrive at the assertion.

Theorem 1. *Let a matrix J have the form* (4) *and its eigenvector $\boldsymbol{y}$ is not proportional to a real vector. Let solutions $f(\zeta)$ and $F(\zeta)$, $\zeta \in D$, of the problem* (16) *be known, where the number $l \in \mathbb{C}$ is* *found by the formula* (16). *Then the function $\omega(z)$ corresponding to the same matrix $J = Q'J_1'(Q')^{-1}$, considered as a solution of the problem* (2), *can be found by the formula*

$$\omega(\zeta) = Q' \cdot (f, g)^T = (\overline{\boldsymbol{y}}, \boldsymbol{y}) \cdot (f, g)^T = \left(f,\ l \cdot \overline{\zeta}\, \frac{df}{d\zeta} + F\right)^T. \tag{17}$$

The converse statement is also valid: If a solution $\omega(\zeta) \in C^1(\overline{D})$ of the problem (2) *exists, then one can construct a solution f, F of the problem* (16).

We note the following important consequence of Eq. (16).

Remark 2. We represent the number $l \in \mathbb{C}$ in (16) in the exponential form: $l = |l|e^{i\xi}$. In (16), we perform the substitution $f = f_1 e^{-i\xi/}$, $F = F_1 e^{i\xi/2}$. Then after cancellation by $e^{i\xi/2}$ the problem (16) takes the following form:

$$|l| \cdot \overline{\zeta}\, \frac{df_1}{d\zeta} + \overline{f}_1 + F_1\big|_\Gamma = 0, \quad f_1, F_1 \in C^1(\overline{D}), \quad |l| = \frac{1}{2}\left|\frac{\det(\overline{\boldsymbol{y}}, \boldsymbol{y})}{\det(\boldsymbol{x}, \boldsymbol{y})}\right|. \tag{18}$$

Hence the problems (16) and (18) are equivalent. Therefore, without loss of generality, we may examine the problem (16) only for $l = |l|$.

3. Construction of two linearly independent solutions of the homogeneous Schwarz problem. First, we construct solutions for a matrix J of the form (4), and then we generalize the result to (2×2)-matrices with an arbitrary multiple eigenvalue λ. Our main result is Theorem 3.

We consider the following complex polynomial $\zeta(z)$ with the real parameter a, where $|a| > 2$:

$$\zeta(z) = z^2 + az = \left(z + \frac{a}{2}\right)^2 - \frac{a^2}{4}, \quad a \in \mathbb{R}, \quad |a| > 2. \tag{19}$$

Proposition 2. *The polynomial $\zeta(z)$ with $|a| > 2$ conformally maps the unit circle K with boundary γ to a certain domain D with the boundary Γ, where $\zeta(\gamma) = \Gamma$.*

Proof. Note that $d\zeta/dz = 2z + a \neq 0$, if $|z| \leq 1$ and $|a| > 2$. The function inverse to (19) has the following form:

$$z(\zeta) = \sqrt{\zeta + \frac{a^2}{4}} - \frac{a}{2}, \quad a \in \mathbb{R}, \quad |a| > 2. \tag{20}$$

The function (20) has a branch point $z_0 = -a^2/4$. One can take any of two branches of the square root. But when determining the selected branch of the root in (20), we assume that $\arg\left(\zeta + a^2/4\right) \in (-\pi, +\pi]$, so that the cut line runs along the ray $(-\infty, z_0]$.

In (19), we perform the substitution $z = e^{it}$, $t \in (-\pi, +\pi]$. Then we obtain the following parametrization of the contour Γ:

$$\Gamma = \zeta(\gamma): \begin{cases} x = \cos 2t + a\cos t, \\ y = \sin 2t + a \sin t, \end{cases} \quad t \in (-\pi, \pi], \quad |a| > 2. \tag{21}$$

We find points where $y(t)$ in (21) vanishes, i.e., solve the equation

$$\sin 2t + a\sin t = 2\sin t\cos t + a\sin t = \sin t(2\cos t + a) = 0.$$

For $|a| > 2$, this equation has only two solutions: $t = \{\pi; 0\} \in (-\pi, \pi]$, i.e., the curve Γ intersects the axis Ox only at two points z_1 and z_2. According to the first equation (21), these points have coordinates $z_1 = x(\pi) = 1 - a$, $z_2 = x(0) = 1 + a$, and for $|a| > 2$ they lie to the right of the branch point $z_0 = -a^2/4$. Thus, Γ does not intersect the ray $(-\infty, z_0]$ containing the cut of the function $\overline{D}$. Therefore, the closure $\overline{D}$ of the domain D does not intersect the ray $(\infty, z_0]$. Hence $\overline{D}$ lies in the domain of holomorphy of the function $z(\zeta)$. $\square$

We state the following assertion, which will be used later.

Proposition 3. *For $a = a_0$ and $a = -a_0$, $a_0 \neq 0$, the formula* (21) *defines the same contour* Γ.

Proof. Let $a = a_0$ and let the vector-valued function (21) take a certain value at $t = t_0$. If $a = -a_0$, then (21) takes the same value at $t = t_0 + \pi$. □

Remark 3. As is well known (see [1]), the interior of a circle cannot be conformally mapped into the interior of an ellipse by an elementary function. Therefore, the contour Γ defined by the formulas (21) is an ellipse.

We use the function (19) for constructing two *linearly independent* solutions of the problem (2) corresponding to a matrix J of the form (4) and defined in the same domain D with boundary Γ given by (21). To do this, by virtue of Theorem 1 and Remark 2, we must find two holomorphic functions f and F in D as solutions of Eq. (16) for some values of the *real* parameter $l = l(J)$.

As above, let K be the unit circle and $\gamma = \partial K$. In (16) we perform the substitutions $\zeta = \zeta(z)$, $f(\zeta) = f(\zeta(z)) = f(z)$, and $F(\zeta) = F(\zeta(z)) = F(z)$, where $\zeta(z)$ has the form (19). We obtain the following functional equation:

$$l \cdot \overline{\zeta}(z) \cdot \frac{df/dz}{d\zeta/dz} + \overline{f}(z) + F(z)\Big|_{\gamma} = 0, \quad \frac{d\zeta}{dz} \neq 0, \quad z \in K = \zeta^{-1}(D), \quad l \in \mathbb{R}.$$

Now we multiply both sides of the last equality by $\partial\zeta/\partial z$:

$$l \cdot \overline{\zeta} \cdot \frac{df}{dz} + \overline{f} \cdot \frac{d\zeta}{dz} + F(z) \cdot \frac{d\zeta}{dz}\Big|_{\gamma} = 0, \quad z \in K, \quad l \in \mathbb{R}. \tag{22}$$

The functions f and F in (22) are defined in the unit circle K with boundary $\gamma = \partial K$. We construct holomorphic functions $f = f(z)$ and $F = F(z)$ as solutions of Eq. (22). Let $c_1, c_2 \in \mathbb{R}$ be indefinite coefficients. In (22), we set

$$\begin{gathered} f(z) = c_1 z + c_2 z^2, \quad \overline{f}(z) = c_1 \overline{z} + c_2 \overline{z}^2, \quad \frac{df}{dz} = c_1 + 2c_2 z, \\ \zeta(z) = z^2 + az, \quad \overline{\zeta}(z) = \overline{z}^2 + a\overline{z}, \quad \frac{d\zeta}{dz} = 2z + a, \quad F = F(z). \end{gathered} \tag{23}$$

Now we perform the substitutions (23) in (22) and set $z = e^{it}$. Then for determining the numbers c_1 and c_2 and the function $F(z)$, we get the following expression:

$$\begin{aligned} l \cdot \left(e^{-2it} + ae^{-it}\right) \cdot (c_1 + 2c_2 e^{it}) + \left(c_1 e^{-it} + c_2 e^{-2it}\right)(2e^{it} + a) \\ + F(e^{it}) \cdot (2e^{it} + a) \equiv 0, \quad t \in (-\pi, \pi], \quad l \in \mathbb{R}. \end{aligned} \tag{24}$$

Equating the coefficients of e^{-it} and e^{-2it} in the first two terms of (24) to zero, we obtain

$$a(l+1)c_1 + 2(l+1)c_2 = 0, \quad lc_1 + ac_2 = 0. \tag{25}$$

Then taking into account (25), we reduce (24) to the following form:

$$2lac_2 + 2c_1 + (2z + a)F(z)\big|_{\gamma} = 0;$$

hence

$$F(z) = -\frac{2(lac_2 + c_1)}{2z + a}, \quad 2z + a \neq 0, \quad |a| > 2, \quad |z| \leq 1. \tag{26}$$

The equalities (25) form a system of linear homogeneous algebraic equations for the real variables c_1 and c_2. The determinant of the system (25) has the form

$$\Delta = (l+1)(a^2 - 2l);$$

therefore, $\Delta = 0$ for $l = -1$ and for $l = a^2/2$.

It is easy to show that for $l = -1$, nontrivial solutions (25) yield the following solutions for (23) and (26):

$$f(z) = c_2\zeta(z), \quad F(z) = 0.$$

In this case,

$$f(z(\zeta)) = c_2\zeta(z(\zeta)) = c_2\zeta.$$

Therefore, using the formula (17), we obtain the linear function $\omega(\zeta)$ as a solution of the problem (2). But a linear function can be a solution to the homogeneous Schwarz problem in a finite domain D only if $\mathrm{Re}\,\omega(\zeta) \equiv 0$. Such solutions are quite trivial, and it makes no sense to study them.

Let us consider the second case:

$$l = \frac{a^2}{2}, \quad a = \pm\sqrt{2l}. \tag{27}$$

Assume that for some matrix J of the form (4), the number l is such that $l = |l| > 2$ (see (16)). Here, for definiteness, we set $a = +\sqrt{2l} > 2$ in (27); then Proposition 2 is valid. Take any nonzero solution (c_1, c_2) of the system (25) for the values of the parameters $l = l$ and $a = +\sqrt{2l}$. Hence we obtain the functions $f(z)$ and $F(z)$ (see (23) and (26)). Next, performing the substitution (20), *due to the invertibility of the transformation of Eq.* (16) *into Eq.* (22), we obtain the functions $f(\zeta)$ and $F(\zeta)$ as solutions of Eq. (16) for $l = |l|$ in the domain D with boundary Γ (see (21)).

Now we assume that for a matrix J of the form (4), the number l is such that $l = |l|e^{i\xi}$, where $\xi \neq 0$ and $|l| > 2$. Then for the values of the parameters $l = |l|$ and $a = +\sqrt{2|l|}$, we also find a solution (c_1, c_2) of the system (25) and, respectively, the functions f_1 and F_1 (see (23) and (26)). Then we perform the substitution (20) and apply the substitutions $f_1 = e^{i\xi/2} \cdot f$ and $F_1 = e^{-i\xi/2} \cdot F$, which are inverse to those made in Remark 2. As a result, we obtain solutions $f(\zeta)$ and $F(\zeta)$ of Eq. (16) for given l. Then, in both cases, using the formula (17) and taking into account Theorem 1, we construct a solution $\omega_a^+(\zeta)$ of the homogeneous Schwarz problem, which is defined in the domain D with boundary Γ (see (21)) and corresponds to thematrix J of the form (4).

Now in (27), we set $a = -\sqrt{2|l|}$. Then, using the scheme described above, we obtain another solution (c_1, c_2) of the system (25) and, respectively, another solution $\omega_a^-(\zeta)$ of the homogeneous Schwarz problem. But at the same time, the number l remains the same. Therefore, the solution of $\omega_a^-(\zeta)$ corresponds to the same matrix J of the form (4). According to Proposition 3, this solution is defined in the same domain D with the boundary Γ (see (21)).

For the solutions constructed, we prove the following assertion .

Proposition 4. *The solutions $\omega_a^+(\zeta)$ and $\omega_a^-(\zeta)$, $\zeta \in D$, of the homogeneous Schwarz problem are linearly independent.*

Proof. On the contrary, assume that $\alpha_1\omega_a^+ + \alpha_2\omega_a^- = 0$, where at least one of the numbers α_1 or α_2 is nonzero. According to (17) we have

$$\omega_a^+ = Q' \cdot (f_1, g_1), \quad \omega_a^- = Q' \cdot (f_2, g_2).$$

Then

$$\alpha_1 Q' \cdot (f_1, g_1)^T + \alpha_2 Q' \cdot (f_2, g_2)^T = 0, \quad |\alpha_1| + |\alpha_2| \neq 0.$$

Multiplying both sides of this relation by the matrix $(Q')^{-1}$ from the left, we obtain the equality

$$\alpha_1 f_1 + \alpha_2 f_2 = 0, \quad |\alpha_1| + |\alpha_2| \neq 0, \tag{28}$$

which means that the functions f_1 and f_2 are linearly dependent. But according to (25), (23), and (20), for $l = l$ and $a = +\sqrt{2|l|}$ we have

$$f_1(\zeta) = z - \frac{\sqrt{2|l|}}{2}\, z^2\Big|_{z=\sqrt{\zeta+\frac{|l|}{2}}-\sqrt{\frac{|l|}{2}}} = (1+|l|)\sqrt{\zeta + \frac{|l|}{2}} - \sqrt{\frac{|l|}{2}}\zeta - (1+|l|)\sqrt{\frac{|l|}{2}}.$$

At the same time, for $l = l$ and $a = -\sqrt{2|l|}$ we have

$$f_2(\zeta) = z + \frac{\sqrt{2|l|}}{2}\, z^2\Big|_{z=\sqrt{\zeta+\frac{|l|}{2}}+\sqrt{\frac{|l|}{2}}} = (1+|l|)\sqrt{\zeta+\frac{|l|}{2}} + \sqrt{\frac{|l|}{2}}\zeta + (1+|l|)\sqrt{\frac{|l|}{2}}.$$

Thus, the functions $f_1(\zeta)$ and $f_2(\zeta)$ are linearly independent, which contradicts (28). The contradiction proves the linear independence of the functions $\omega_a^+(\zeta)$ and $\omega_a^-(\zeta)$. □

As a result, taking into account Propositions 3 and 4, Theorem 1, and Remark 2, we arrive at the following theorem.

Theorem 2. *Let a matrix J of the form* (4) *be such that $|l(J)| > 2$ in* (16). *Moreover, assume that its eigenvector $\boldsymbol{y}$ is not proportional to a real vecor. Then in the domain D bounded by a contour Γ of the form* (21), *there exist two linearly independent solutions $\omega_a^+(\zeta)$ and $\omega_a^-(\zeta)$ of the homogeneous Schwarz problem* (2), *which correspond to the matrix J and the values of the parameter $a = \pm\sqrt{2|l|}$.*

Now we summarize the results. Let a matrix $J_\lambda \in \mathbb{C}^{2\times 2}$ have a multiple eigenvalue $\lambda = \lambda_1 + \lambda_2 i$, where $\lambda_2 \neq 0$, and the corresponding eigenvector $\boldsymbol{y}$ is not proportional to a real vector. As above, we denote by $Q = (\boldsymbol{x}, \boldsymbol{y})$ a Jordan basis of this matrix. For the matrix J_λ, the number $l_1 = l_1(J_\lambda)$ is defined by the same formula (5). Therefore, $l = l(J_\lambda)$ is defined by the formula (16).

Remark 4. Proposition 1 is also valid for matrices with an arbitrary multiple eigenvalue λ since its proof was based only on the Jordan form of the matrix J.

Proposition 5. *Let a matrix J have the form* (4). *We introduce the notation $J_\lambda = \lambda_1 E + \lambda_2 J$, where $\lambda_2 \neq 0$. Then, taking into account the notation* (5) *and* (16), *the following equalities hold:*

$$|l_1(J_\lambda)| = |\lambda_2 \cdot l_1(J)|, \quad |l(J_\lambda)| = |\lambda_2 \cdot l(J)|. \tag{29}$$

Proof. Taking into account Remark 4, we apply Proposition 1. We choose Jordan bases of the matrices J and J_λ. We take a vector $\boldsymbol{x} \neq 0$ such that

$$(J - iE)\boldsymbol{x} \neq 0, \quad (J_\lambda - \lambda E)\boldsymbol{x} \neq 0.$$

Then

$$Q_\lambda = (\boldsymbol{x}, \boldsymbol{y}_1) = \big(\boldsymbol{x},\, (J_\lambda - \lambda E)\boldsymbol{x}\big)$$

is a Jordan basis of J_λ. Note that

$$J_\lambda - \lambda E = \lambda_1 E + \lambda_2 J - \lambda_1 E - \lambda_2 i E = \lambda_2 (J - iE).$$

Moreover, $Q = (\boldsymbol{x}, \boldsymbol{y}_2) = \big(\boldsymbol{x},\, (J - iE)\boldsymbol{x}\big)$ is a Jordan basis for J. Thus, the generalized eigenvector $\boldsymbol{x}$ in both cases is the same, while the eigenvectors are related by the relation $\boldsymbol{y}_1 = \lambda_2 \boldsymbol{y}_2$. Substituting this equality into (5) and taking into account (16) and Proposition 1, we obtain (29). □

As above, assume that the matrix $J_\lambda \in \mathbb{C}^{2\times 2}$ has a multiple eigenvalue λ. Then the matrix $J_\lambda \in \mathbb{C}^{2\times 2}$ has a multiple eigenvalue i. Let the conditions of Theorem 2 be fulfilled for J, i.e., $|l(J)| > 2$. By virtue of Proposition 5, this inequality is equivalent to the inequality $|l(J_\lambda)| > 2|\lambda_2|$. In the solutions $\omega_a^\pm(\zeta)$ of the homogeneous Schwarz problem corresponding to the matrix J, we perform the following invertible substitution:

$$x = x' + \lambda_1 y', \quad y = \lambda_2 y', \quad \lambda_2 \neq 0. \tag{30}$$

It is easy to show that after the substitution (30), the functions $\omega_a^\pm(x', y')$ become J-analytic functions with the matrix J_λ. Obviously, they remain linearly independent. After the substitution (30) into (21), we see that the pair of functions $\omega_a^{\pm}(x', y')$ is a solution of the problem (2) in the domain D_λ bounded by the following contour $\Gamma_\lambda = \partial D_\lambda$:

$$\Gamma_\lambda : \begin{cases} x' = \cos 2t + a\cos t - \dfrac{\lambda_1}{\lambda_2}(\sin 2t + a \sin t), \\ y' = \dfrac{1}{\lambda_2}(\sin 2t + a\sin t), \end{cases} \qquad t \in (-\pi, \pi], \quad |a| > 2, \quad \lambda_2 \neq 0. \tag{31}$$

As a result, taking into account Theorem 2 and (29), we arrive at the following theorem, which is the main result of the present paper.

Theorem 3. *Let a matrix J_λ have a multiple eigenvalue $\lambda = \lambda_1 + \lambda_2 i$, $\lambda_2 \neq 0$, and the corresponding eigenvector is not proportional to a real vector. Moreover, let $|l| = |l(J_\lambda)| > 2|\lambda_2|$ in the formula* (16). *Then in the domain D bounded by the contour Γ* (*see* (31)), *there exist two linearly independent solutions $\omega_a^+(x', y')$ and $\omega_a^-(x', y')$ of the homogeneous Schwarz problem* (2), *which correspond to the matrix J_λ and the values of the parameter $a = \pm\sqrt{|2l/\lambda_2|}$.*

As an example, we construct two functions $\omega_a^+(\zeta)$ and $\omega_a^-(\zeta)$, which correspond to the matrix J of the form (3) considered above with a multiple eigenvalue $\lambda = i$. The Jordan form J_1 and a Jordan basis Q (defined nonuniquely) of the matrix (3) are as follows:

$$J_1 = \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix}, \quad Q = (\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} 1 & 3i \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 4i & 9 \\ 1 & -2i \end{pmatrix} = Q J_1 Q^{-1}. \tag{32}$$

In (32), the eigenvector $\boldsymbol{y} = (J - iE)\cdot \boldsymbol{x} = (3i, 1)$ of the matrix J is not proportional to a real vector. According to (16), we have $l = l(J) = 3 > 2$, so we can apply Theorem 2 for J. According to (17), we find

$$\omega_a^{\pm}(\zeta) = (\overline{\boldsymbol{y}}, \boldsymbol{y}) \cdot \left(f,\ 3\overline{\zeta}\frac{df}{d\zeta} + F\right)^T = \begin{pmatrix} -3i & 3i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} f \\ 3\overline{\zeta}\dfrac{df}{d\zeta} + F \end{pmatrix} = \begin{pmatrix} 9i\overline{\zeta}\dfrac{df}{d\zeta} - 3if + 3iF \\ 3\overline{\zeta}\dfrac{df}{d\zeta} + f + F \end{pmatrix}. \tag{33}$$

Both functions $\omega_a^+(\zeta)$ and $\omega_a^-(\zeta)$ are calculated by the same formula (33). However, obviously, the functions $f(\zeta) = f(z(\zeta))$ and $F(\zeta) = F(z(\zeta))$ in (33) for $a = +\sqrt{2l}$ and $a = -\sqrt{2l}$ are different; we find them by the formulas (25), (23), (26), (27), and (20). We write these functions for $\omega_a^+(\zeta)$ in (33):

$$l = 3, \quad a = +\sqrt{2l} = \sqrt{6}, \quad c_1 = 1, \quad c_2 = -\frac{\sqrt{6}}{2},$$

$$f(z) = z - \frac{\sqrt{6}}{2}z^2, \quad F(z) = \frac{16}{2z + \sqrt{6}}, \quad z(\zeta) = \sqrt{\zeta + \frac{3}{2}} - \frac{\sqrt{6}}{2}, \quad \zeta \in D,$$

and for $\omega_a^-(\zeta)$ in (33):

$$l = 3, \quad a = -\sqrt{2l} = -\sqrt{6}, \quad c_1 = 1, \quad c_2 = \frac{\sqrt{6}}{2},$$

$$f(z) = z + \frac{\sqrt{6}}{2}z^2, \quad F(z) = \frac{16}{2z - \sqrt{6}}, \quad z(\zeta) = \sqrt{\zeta + \frac{3}{2}} + \frac{\sqrt{6}}{2}, \quad \zeta \in D.$$

In this case, according to the formula (21), taking into account Proposition 3, we conclude that the contour of $\Gamma = \partial D$ has the following parametrization:

$$\Gamma : \begin{cases} x = \cos 2t + \sqrt{6}\cos t, \\ y = \sin 2t + \sqrt{6}\sin t, \end{cases} \qquad t \in [0, 2\pi).$$

The fact that the function (33) corresponds to a matrix J of the form (32) is verified by substitution in (1). The equality $\operatorname{Re}\omega(\zeta)\big|_\Gamma = 0$, where $\Gamma = \zeta(\gamma)$, for (33) is not as obvious as for vector quadratic forms. We prove this equality. According to the algorithm described above with $l = 3$, the functions f and F in (33) are found from the condition

$$3\,\overline{\zeta}\,\frac{df}{d\zeta} + \overline{f} + F\big|_{\Gamma} = 0,$$

from which

$$F\big|_{\Gamma} = -3\,\overline{\zeta}\,\frac{df}{d\zeta} - \overline{f}. \tag{34}$$

Substituting (34) into (33) we obtain the equality

$$\omega_a^{\pm}(\zeta)\big|_{\Gamma} = \big(-6i\operatorname{Re} f,\ 2i\operatorname{Im} f\big),$$

whence $\operatorname{Re}\omega_a^{\pm}(\zeta)\big|_{\Gamma} = 0$, which proves the statement.

4. Relationship between the parameter l_1 and the number t_J. Let us consider an interesting generalization of the formula (5). Let $\lambda = \lambda_1 + \lambda_2 i$, $\lambda_2 \neq 0$. We introduce the notation

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{21} \neq 0, \quad t_J = \frac{|\lambda_2 \cdot a_{21}|}{\Big|\operatorname{Im}\big[\overline{a}_{21}(a_{11} - \lambda)\big]\Big|} \in \mathbb{R}. \tag{35}$$

The following assertion was proved in [2].

Theorem 4. *Let a nontriangular matrix $J \in \mathbb{C}^{2\times 2}$ of the form* (35) *have a multiple eigenvalue $\lambda = \lambda_1 + \lambda_2 i$, $\lambda_2 \neq 0$, and let the corresponding eigenvector be not proportional to a real vector. Then the existence of the corresponding solutions of the problem* (2) *in the form of quadratic vector-forms is equivalent to the condition $0 < t_J < 1$ in* (35).

In particular, for the matrix J of the form (3), we have $t_J = 1/3$.

The number $l_1(J)$ defined by the formula (5) was not considered in [2]. We prove the following assertion.

Proposition 6. *The number t_J from* (35) *and the number l_1 from* (5) *are related by the formula*

$$t_J = \frac{2|\lambda_2|}{|l_1|}. \tag{36}$$

Proof. Based on Remark 4, we use Proposition 1. To calculate the absolute value $|l_1|$ of the number l_1 (see (5)), we choose the following Jordan basis Q of the matrix J:

$$\boldsymbol{x} = (1, 0), \quad \boldsymbol{y} = (J - \lambda E)\cdot \boldsymbol{x} = (a_{11} - \lambda,\ a_{21}), \quad a_{21} \neq 0, \quad Q = (\boldsymbol{x}, \boldsymbol{y}). \tag{37}$$

According to the formula (5), taking into account the notation (35), we have

$$\begin{aligned} l_1 &= \frac{\det(\overline{\boldsymbol{y}}, \boldsymbol{y})}{\det(\boldsymbol{x}, \boldsymbol{y})} = \frac{\begin{vmatrix} \overline{a_{11} - \lambda} & a_{11} - \lambda \\ \overline{a}_{21} & a_{21} \end{vmatrix}}{\begin{vmatrix} 1 & a_{11} - \lambda \\ 0 & a_{21} \end{vmatrix}} = \frac{a_{21}\cdot(\overline{a_{11} - \lambda}) - \overline{a}_{21}\cdot(a_{11} - \lambda)}{a_{21}} \\ &= \frac{a_{21}\cdot(\overline{a_{11} - \lambda}) - \overline{a_{21}\cdot(\overline{a_{11} - \lambda})}}{a_{21}} = \frac{2\operatorname{Im}\big[a_{21}\cdot(\overline{a_{11} - \lambda})\big]\cdot i}{a_{21}} = \frac{-2\operatorname{Im}\big[\overline{a}_{21}\cdot(a_{11} - \lambda)\big]\cdot i}{a_{21}}. \end{aligned} \tag{38}$$

From (38), taking into account (5) and (35), we obtain (36). □

By virtue of Proposition 6, we reformulate Theorem 4 as follows.

Theorem 5. *Let a nontriangular matrix* $J \in \mathbb{C}^{2\times 2}$ *have a multiple eigenvalue* $\lambda = \lambda_1 + \lambda_2 i$, $\lambda_2 \neq 0$, *and let the corresponding eigenvector be not proportional to a real vector. Then the existence of the corresponding solutions of the problem* (2) *in the form of quadratic vector-forms is equivalent to the condition* $|l_1(J)| > 2|\lambda_2|$ *in* (5).

5. Conclusion. Instead of the unit circle, one could consider a circle of arbitrary radius r for constructing the functions $\omega_a^{\pm}(\zeta)$, $\zeta \in D$, for $\lambda = i$. However, simple calculations show that this will not lead to an improvement in the estimate for (i.e., the decreasing the value of $|l|$) in Theorems 2 and 3. Moreover, we note the following fact: in Theorem 3, we have $|l(J_\lambda)| > 2|\lambda_2|$, i.e., according to (16), we have $|l_1(J_\lambda)| > 4|\lambda_2|$. Therefore, by virtue of Theorem 5, for all matrices satisfying the conditions of Theorem 3, solutions of the problem (2) in the form of quadratic vector-forms are also possible. Of course, they can be defined on ellipses.

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